CALC OF INTEGRALS

FUNDAMENTAL PRIMITIVES

Trigonometric functions

1) $\cos x \longrightarrow \sin x + C$
2) $\sin x \longrightarrow -\cos x + C$
3) $\dfrac{1}{\cos^2 x} = 1 + tg^2 x \longrightarrow tg\, x + C$

Iperbolic functions

4) $\sinh x \longrightarrow \cosh x + C$
5) $\cosh x \longrightarrow \sinh x + C$

Inverse trigonometric functions

6) $\dfrac{1}{\sqrt{1-x^2}} \longrightarrow \arcsin x + C$
7) $\dfrac{1}{1+x^2} \longrightarrow \operatorname{arctg} x + C$

Polynomial and exponential functions

8) $x^\alpha \longrightarrow \dfrac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1 \lor \log x + C, \alpha = -1$
9) $e^x \longrightarrow e^x + C$
10) $b^x = e^{x \log b} \longrightarrow \dfrac{b^x}{\log b} + C$

VERY IMPORTANT REMINDERS

Trigonometric

$\boxed{\cos^2 x + \sin^2 x = 1}$

$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x \overset{!}{=} 1 - 2\sin^2 x \overset{!}{=} 2\cos^2 x - 1$

Iperbolic

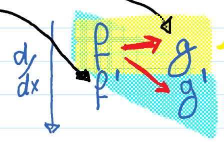
$\boxed{\cosh^2 x - \sinh^2 x = 1}$

$\sinh 2x = 2 \sinh x \cosh x$
$\cosh 2x = \cosh^2 + \sinh^2 x \overset{!}{=} 1 + 2\sinh^2 x \overset{!}{=} 2\cosh^2 x - 1$

GENERAL METHODS

1) CHAIN RULE $\int g'(f(x))\, f'(x)\, dx = g(f(x)) + C$

2) INT. BY PARTS $\int f'(x)\, g(x)\, dx = f(x) g(x) - \int f(x)\, g'(x)\, dx$



3) SUBSTITUTION $\int f(x)\, dx = \int f(g(t))\, g'(t)\, dt$

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$x = g(t)$

$dx = g'(t)\,dt$

STRATEGIES FOR PARTICULAR INTEGRALS

Integrals with tangent / cotangent

1) $\int tg\,x\,dx = \int \underbrace{\frac{\sin x}{\cos x}}\,dx = -\log(\cos x) + C$

$\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c$ [this is a general and very useful strategy]

2) $\int \frac{1}{\sin^2 x}\,dx = \int \left(\frac{\cos x}{\sin x}\right)^2 \frac{1}{\cos^2 x}\,dx = \int \underbrace{tg^{-2} x}_{f(x)^{-2}} \cdot \underbrace{\frac{1}{\cos^2 x}}_{f'(x)}\,dx = -tg^{-1} x + C = -\frac{\cos x}{\sin x} + C$

we can also remember the fundamental integral

$\frac{1}{\sin^2 x} = 1 + ctg^2 x \longrightarrow -ctg\,x + C$ $\quad ctg\,x = \frac{1}{tg\,x} = \frac{\cos x}{\sin x}$

Integrals of trigonometric powers

3) $\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{1}{2}(x + \sin x \cos x) + C$

even $\Rightarrow$ use double angle formula to obtain lower order terms

4) $\int \cos^3 x\,dx = \int \cos x(1-\sin^2 x)\,dx = \int \cos x\,dx - \int \sin^2 x \cos x\,dx = \sin x - \frac{\sin^3 x}{3} + c$

odd $\Rightarrow$ use identity $\cos^2 x + \sin^2 x = 1$ to split the product

Exercises : $\int \sin^2 x\,dx$, $\int \sin^5 x\,dx$

Integrals by parts (polynomial – log / exp / sin / ...)

5) $\int \log x\,dx = \int \underset{f'}{1} \cdot \underset{g}{\log x}\,dx$ $\left[\begin{array}{cc} f & g \\ x & \log x \\ 1 & \frac{1}{x} \end{array}\right]$ ($\frac{d}{dx}$) $= x\log x - \int dx = x(\log x - 1) + C$

6) $\int x e^x\,dx = \int \underset{g}{x} \cdot \underset{f'}{e^x}\,dx$ $\left[\begin{array}{cc} f & g \\ e^x & x \\ e^x & 1 \end{array}\right] = x e^x - \int e^x\,dx = e^x(x-1) + c$

7) $\int x \sin x\,dx = \int \underset{g}{x} \cdot \underset{f'}{\sin x}\,dx$ $\left[\begin{array}{cc} f & g \\ -\cos x & x \\ \sin x & 1 \end{array}\right] = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$

Exercises : $\int x \log x\,dx$, $\int x^2 e^x\,dx$, $\int x^2 \cos x\,dx$

"Cyclic" integrals

"Cyclic" integrals

8) $\int e^x \sin x\,dx = \begin{bmatrix} f & g \\ e^x & \sin x \\ e^x & \cos x \end{bmatrix} = e^x \sin x - \int e^x \cos x\,dx \begin{bmatrix} f & g \\ e^x & \cos x \\ e^x & -\sin x \end{bmatrix}$

$= e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x)\,dx \right] = e^x(\sin x - \cos x) - \int e^x \sin x\,dx$

$\Rightarrow 2\int e^x \sin x\,dx = e^x(\sin x - \cos x) \Rightarrow \int e^x \sin x\,dx = \frac{1}{2} e^x(\sin x - \cos x) + C$

Exercises: $\int e^{2x}\cos x\,dx$, $\int \cos 3x \cos x\,dx$

Integrals with substitutions

9) $\int \arcsin x\,dx \underset{\substack{x=\sin y \\ dx = \cos y\,dy}}{=} \int \underset{g}{y}\,\underset{f'}{\cos y}\,dy \begin{bmatrix} f & g \\ \sin y & y \\ \cos y & 1 \end{bmatrix} = y\sin y - \int \sin y\,dy = y \sin y + \cos y + C$

$= y \sin y + \sqrt{1-\sin^2 y} + C = x \arcsin x + \sqrt{1-x^2} + C$

Remember to substitute back at the end!

Exercises: $\int \sqrt{e^x - 1}\,dx$, $\int \frac{1}{x+\sqrt{x}}\,dx$, $\int \sin \log x\,dx$

Integrals with iperbolic functions

10) $\int \cosh^2 x\,dx = \int \frac{1}{2}(1+\cosh 2x)\,dx = \frac{1}{2}(x + \sinh x \cosh x) + C$

11) $\int \sinh^2 x\,dx = \int \frac{1}{2}(-1+\cosh 2x)\,dx = \frac{1}{2}(-x + \sinh x \cosh x) + C$

NOTE: $\operatorname{arcsinh} x = y$

we will use it in a moment!

$x = \sinh y = \frac{e^y - e^{-y}}{2}$ $\qquad e^y - 2x - e^{-y} = 0$

$e^{-y}(e^{2y} - 2x e^y - 1) = 0 \overset{e^y = t}{\Leftrightarrow} t^2 - 2xt - 1 = 0$ with $t = e^y$

$t_{1,2} = x \pm \sqrt{x^2+1} \Leftrightarrow e^y = x \pm \sqrt{x^2+1}$

1) $\boxed{y = \ln(x + \sqrt{x^2+1})}$

2) $x - \sqrt{x^2+1} < 0$ no solutions

Exercise: simplify $\operatorname{arccosh} x = y$ $\qquad [y = \ln(x + \sqrt{x^2-1})]$

Integrals with $\sqrt{\pm x^2 \pm \alpha^2}$

12) $\int \sqrt{1-x^2}\,dx \underset{\substack{x=\sin y \\ dx=\cos y\,dy}}{=} \int \sqrt{1-\sin^2 y}\,\cos y\,dy = \int \cos^2 y\,dy \overset{*}{=} \frac{1}{2}(y + \sin y \cos y) + C$

already solved

$= \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C$

Verify for exercise

Using the sin we can rewrite $\sqrt{1-\sin^2} = \cos$
and obtain $\int \cos^2$ that we already solve!
At the end, we substitute back $x = s$.

Using the sin we can rewrite $\sqrt{1-\sin^2}=\cos$ and obtain $\int \cos^2$ that we already solve! At the end, we substitute back $x = s$.

↳ Verify for exercise

13) $\int \sqrt{1+x^2}\,dx \underset{\substack{x=\sinh y \\ dx=\cosh y}}{=} \int \sqrt{1+\sinh^2 y}\,\cosh y\,dy = \int \cosh^2 y\,dy \overset{*}{=} \frac{1}{2}(y+\sinh y\cosh y)+c$

$= \frac{1}{2}\left(\operatorname{arcsinh} x + x\sqrt{1+x^2}\right)+c$

$= \frac{1}{2}\left(\ln\left(x+\sqrt{1+x^2}\right)+x\sqrt{1+x^2}\right)+c$

This time we use sinh to rewrite $\sqrt{1-\sinh^2}=\cosh$ and we use the semplification of arcsinh x shown before.

↳ Verify for exercise

14) $\int \sqrt{x^2-1}\,dx \underset{\substack{x=\cosh y \\ dx=\sinh y\,dy}}{=} \int \sqrt{\cosh^2 y-1}\,\sinh y\,dy = \int \sinh^2 y\,dy \overset{*}{=} \frac{1}{2}(-y+\sinh y\cosh y)+c$

$= \frac{1}{2}\left(-\operatorname{arccosh} x + x\sqrt{x^2-1}\right)+c$

$= \frac{1}{2}\left(-\ln\left(x+\sqrt{x^2-1}\right)+x\sqrt{x^2-1}\right)+c$

Same as above but with $\sqrt{\cosh^2-1}=\sinh$ and semplification for arccosh.

↳ Verify for exercise

Exercises: $\int \frac{1}{\sqrt{1-x^2}}dx$, $\int \frac{1}{\sqrt{1+x^2}}dx$, $\int \frac{1}{\sqrt{x^2-1}}dx$

Integrals of fractions

15) $\int \frac{n(x)}{d(x)}dx$ with $\deg(n(x))<\deg(d(x))$ $= \int \frac{A}{ax+b}+\frac{A'}{(ax+b)^2}+\dots+\frac{Bx+C}{\underbrace{ax^2+bx+c}_{\text{with } \Delta<0}}+\frac{B'x+C'}{(ax^2+bx+c)^2}+\dots$

↳

$$\begin{cases}
\int \frac{1}{ax+b}dx = \frac{1}{a}\log(ax+b)+c \\
\int \frac{1}{(ax+b)^n}dx = \frac{1}{a(1-n)}(ax+b)^{1-n}+c, \quad n\neq 1 \\
\int \frac{x}{x^2+b}dx = \frac{1}{2}\log(x^2+b)+C, \quad b \gtrless 0 \\
\int \frac{1}{x^2-b^2}dx = \int \frac{1}{(x-b)(x+b)}dx = \int \frac{dx}{x-b}-\int\frac{dx}{x+b} = \log(x-b)-\log(x+b)+c = \log\left(\frac{x-b}{x+b}\right)+c \\
\int \frac{1}{x^2+b^2}dx = \frac{1}{b}\operatorname{arctg}\frac{x}{b}+c
\end{cases}$$

NOTE: to decompose the fraction you start from the result and find the values of A, B, ... with a system of equations

e.g. $\frac{1}{x(1+x)^2} = \frac{A}{x}+\frac{B}{1+x}+\frac{C}{(1+x)^2} = \frac{A(1+x)^2+Bx(1+x)+C(x)}{x(1+x)(1+x)^2}$

$\Rightarrow 0\cdot x^2+0\cdot x+1 = Ax^2+A+Bx^2+Bx+Cx$

$$\begin{array}{l} x^2 \\ x \\ 1 \end{array}\begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = A \end{cases} \rightarrow \begin{cases} A=1 \\ B=-1 \\ C=1 \end{cases} \rightarrow \frac{1}{x}-\frac{1}{1+x}+\frac{1}{(1+x)^2}$$

NOTE: when you have $\frac{1}{ax^2+bx+c}$ with $\Delta<0$
you can always "complete the square":
$c = c' + \gamma$ s.t. ax^2+bx+c' is a square $(\alpha x+\beta)^2$
It means that we can always write
$ax^2+bx+c = (\alpha x+\beta)^2+\gamma$ with $\gamma \geq 0$
thus we are in the cases $\frac{1}{f^2(x) \pm b^2}$

e.g. $x^2+5x+10$ $\Delta = 25-40<0$

$$\Rightarrow \begin{cases} x^2 \to x^2 \\ 5x \to 2\cdot\left(\frac{5}{2}\right)(x) \\ 10 \to \left(\frac{5}{2}\right)^2 + \frac{35}{4} \end{cases} \Rightarrow \left(x+\frac{5}{2}\right)^2 + \left(\frac{\sqrt{35}}{2}\right)^2$$

NOTE: when $\deg(n(x)) \geq \deg(d(x))$ we divide $n(x)$ by $d(x)$ obtaining a polynomial and the reminder

e.g. $\frac{x^2}{x+1} \Rightarrow$

$$\begin{array}{r|l} x^2+0x+0 & x+1 \\ \underline{x^2+x} & \overline{x-1} \\ // \; -x & \\ \underline{-x-1} & \\ // \; +1 & \end{array} \Rightarrow x-1+\frac{1}{x+1}$$

Exercises: $\int \frac{1}{x(1+x)^2}dx$, $\int \frac{1}{x^2+5x+10}dx$, $\int \frac{x^2}{x+1}dx$

EXERCISES

5) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

6) $\int \frac{1}{x\sqrt{2x-1}}dx$

7) $\int \sqrt{a^2-x^2}\,dx$

8) $\int \frac{1}{x\sqrt{1-\log^2 x}}dx$

9) $\int e^x \log(e^x+1)\,dx$

10) $\int \frac{\sqrt{x^3}}{1+x}dx$

11) $\int \frac{3x+1}{x^2-5x+6}dx$

12) $\int \frac{5x+9}{x^2+2x+5}dx$

13) $\int \frac{1}{\sin^2 x\cos^2 x}dx$

14) $\int x^2\log^2 x\,dx$

15) $\int \frac{1}{x\log^3 x}dx$

16) $\int \frac{\sin 2x}{1+\sin^2 x}dx$

17) $\int \frac{x^3+x^2-x}{x^2+x-6}dx$

18) $\int \frac{1}{\sin x}dx$

19) $\int \frac{1}{\sqrt{1+2x-x^2}}dx$

20) $\int \cos^2 2x \sin x\,dx$