Extra Material Presented in Class

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1 Ad, ad and the Lie bracket

Let G be a Lie group and recall that for all $g \in G$ the conjugation mapping $C_g : G \to G$, $C_g(h) = ghg^{-1}$ is a Lie group isomorphism. Combining [2, Propositions 1.13 and 1.18] we conclude that $\mathrm{Ad}_g := T_e C_g$ is a Lie algebra isomorphism and we have the following commuting diagram.

$$\begin{tikzcd} G \arrow[r, "C_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}_g"'] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

In other words, we have $g\exp(\xi)g^{-1} = \exp(\mathrm{Ad}_g\xi)$ for all $g \in G$, $\xi \in \mathfrak{g}$.

We are now consider the map $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$. Given that Ad_g is an invertible linear map in $\mathfrak{g}$ for all $g \in G$ (because C_g is a diffeomorphism) we conclude that Ad is a mapping from G into $GL(\mathfrak{g})$.

Proposition 1.1. *The map* $\mathrm{Ad} : G \to GL(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$, *is a Lie group homomorphism.*

Proof. Smoothness is inherited from the smoothness of the group operations in G. To show that it is a group homomorphism we differentiate $C_{gh} = C_g \circ C_h$ at the group identity and use the chain rule to obtain:

$$\mathrm{Ad}_{gh} = T_e C_{gh} = T_e(C_g \circ C_h) = T_e C_g \circ T_e C_h = \mathrm{Ad}_g \circ \mathrm{Ad}_h.$$

□

Definition 1.2. Let G be a Lie group and V be a vector space. A Lie group homomorphism $\Phi : G \to GL(V)$ is called a ***representation*** of the Lie group G on the vector space V.

In accordance with the above definition one refers to $\mathrm{Ad} : G \to \mathfrak{g}$ as the *adjoint representation* of the Lie group G.

Recall that for a vector space V, the Lie algebra of $GL(V)$ is $\mathfrak{gl}(V) = L(V)$, the space of linear endomorphisms of V. We now use [2, Propositions 1.13 and 1.18] and differentiate Ad at the identity $e \in G$ to obtain a Lie algebra homomorphism $T_e\mathrm{Ad} : \mathfrak{g} \to L(V)$. We denote $\mathrm{ad} := T_e\mathrm{Ad}$, and for $\xi \in \mathfrak{g}$ write $\mathrm{ad}(\xi) = \mathrm{ad}_\xi \in L(V)$. By its definition, if $\eta \in \mathfrak{g}$ we have

$$\mathrm{ad}_\xi(\eta) = T_e\mathrm{Ad}(\xi)(\eta) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(\xi t)}(\eta).$$

By [2, Proposition 1.18] we have the commuting diagram

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & L(\mathfrak{g}) \arrow[u, "\mathrm{e}"'] \end{tikzcd}$$

where we recall that $\mathrm{e} : L(V) \to GL(V)$ is the operator exponential defined by $\mathrm{e}^f := id_V + f + \frac{1}{2} f \circ f + \frac{1}{3!} f \circ f \circ f + \dots$ for $f \in L(V)$ (which coincides with the Lie group exponential $\exp_{GL(V)}$). In other words, we have the following identity between elements of $GL(\mathfrak{g})$:

$$\mathrm{Ad}_{\exp \xi} = \mathrm{e}^{\mathrm{ad}_\xi} \quad \forall \xi \in \mathfrak{g}.$$

Proposition 1.3. *For $\xi, \eta \in \mathfrak{g}$ we have $\mathrm{ad}_\xi(\eta) = [\![\xi, \eta]\!]$.*

Proof. By definition of the Lie bracket ([2, Section 1.2.D]) we have

$$[\![\xi, \eta]\!] = [X_\xi, X_\eta](e) = L_{X_\xi} X_\eta(e) = \frac{d}{dt}\bigg|_{t=0} (\Phi_t^{X_\xi})^* X_\eta(e).$$

Now, in order to compute the pull-back $(\Phi_t^{X_\xi})^* X_\eta$ recall that for all $g \in G$ we have

$$\Phi_t^{X_\xi}(g) = g\Phi_t^{X_\xi}(e) = g \exp(t\xi) = R_{\exp(t\xi)}(g).$$

It then follows that

$$\Phi_t^{X_\xi} = R_{\exp(t\xi)}, \qquad (\Phi_t^{X_\xi})^{-1} = R_{\exp(-t\xi)}.$$

Now recall that if $\Psi : M \to M$ is a diffeomorphism and $Y \in \mathfrak{X}(M)$ then $\Psi^* Y(m) = T_{\Psi(m)} \Psi^{-1}(\Psi(m))$ for all $m \in M$. Therefore,

$$(\Phi_t^{X_\xi})^* X_\eta(e) = (R_{\exp(t\xi)})^* X_\eta(e) = T_{\exp(t\xi)} R_{\exp(-t\xi)} X_\eta(\exp(t\xi)).$$

But, by left invariance of X_η we have $X_\eta(\exp(t\xi)) = T_e L_{\exp(t\xi)}(\eta)$, so we may write

$$\begin{aligned} (\Phi_t^{X_\xi})^* X_\eta(e) &= T_{\exp(t\xi)} R_{\exp(-t\xi)} \circ T_e L_{\exp(t\xi)}(\eta) \\ &= T_e (R_{\exp(-t\xi)} \circ L_{\exp(t\xi)})(\eta) \\ &= \mathrm{Ad}_{\exp(t\xi)}(\eta). \end{aligned}$$

Hence, combining the above identities we obtain:

$$[\![\xi, \eta]\!] = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}_{\exp(t\xi)}(\eta)) = \mathrm{ad}_\xi(\eta).$$

□

Remark 1.4. The above proposition provides us with an alternative approach to compute the Lie bracket on the Lie algebra. Moreover, it shows that the Lie bracket can be obtained by performing two derivatives in the following way. If $t \mapsto g(t)$ and $s \mapsto h(s)$ are curves on G which satisfy $g(0) = h(0) = e$ and $g'(0) = \xi$, $h'(0) = \eta$ with $\xi, \eta \in \mathfrak{g}$ then we have

$$\frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} g(t)h(s)g(t)^{-1} = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{g(t)}(\eta) = \mathrm{ad}_\xi \eta = [\![\xi, \eta]\!].$$

As an immediate consequence of this calculation one can give a proof that the Lie algebra of an abelian Lie group is abelian.

Example. Let us apply the above remark to compute the Lie algebra bracket on the Lie algebra of the Lie group $GL(n,\mathbb{R})$. As we know, the Lie algebra is $L(n,\mathbb{R})$ and the exponential map $\exp_{GL(n,\mathbb{R})}$ is the usual matrix exponential $\exp_{GL(n,\mathbb{R})} = \mathrm{e} : L(n,\mathbb{R}) \to GL(n,\mathbb{R})$, $\xi \mapsto \mathrm{e}^{\xi}$.

Let $A \in GL(n,\mathbb{R})$ and $\eta \in L(n,\mathbb{R})$. We compute

$$\mathrm{Ad}_A(\eta) = \frac{d}{ds}\bigg|_{s=0} A\mathrm{e}^{s\eta}A^{-1} = A\eta A^{-1}.$$

Now we set $A = \mathrm{e}^{\xi t}$ for some $\xi \in L(n,\mathbb{R})$ and compute

$$[\![\xi,\eta]\!] = \mathrm{ad}_{\xi}(\eta) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\mathrm{e}^{\xi t}}(\eta) = \frac{d}{dt}\bigg|_{t=0} \mathrm{e}^{\xi t}\eta\mathrm{e}^{-\xi t} = \xi\eta - \eta\xi.$$

This shows that the Lie bracket of the Lie algebra $GL(n,\mathbb{R})$ coincides with the matrix commutator in $L(n,\mathbb{R})$. (An alternative proof of this fact, that relies on the definition of the bracket in terms of commutators of left invariant vector fields is given in [2, Section 1.2.D]).

A similar calculation shows that the Lie bracket of the Lie algebra of $GL(n,\mathbb{C})$ (or, more generally, of $GL(V)$ with V a vector space) is the matrix (operator) commutator in $L(n,\mathbb{C})$ ($L(V)$).

Having understood what the Lie bracket of the Lie algebra of $GL(\mathfrak{g})$ is, we come back to the mapping $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g})$. Considering that $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g})$ is a Lie algebra homomorphism we obtain the formula

$$\mathrm{ad}_{[\![\xi,\eta]\!]} = \mathrm{ad}_{\xi} \circ \mathrm{ad}_{\eta} - \mathrm{ad}_{\eta} \circ \mathrm{ad}_{\xi} \qquad \forall \xi,\eta \in \mathfrak{g}.$$

It is a straightforward calculation to check that the above formula acting on an arbitrary element $\zeta \in \mathfrak{g}$, together with Proposition 1.3, recovers the Jacobi identity.

The Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g})$ is called the *adjoint representation* of the Lie algebra $\mathfrak{g}$. This terminology comes from the following definition:

Definition 1.5. Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. A Lie algebra homomorphism $\phi : \mathfrak{g} \to L(V)$ is called a ***representation*** of the Lie algebra $\mathfrak{g}$ in the vector space V.

Using [2, Proposition 1.13] we conclude that if $\Phi : G \to GL(V)$ is a Lie group representation of the Lie group G on the vector space V, then $\phi = T_{e_G}\Phi : \mathfrak{g} \to L(V)$ is a Lie algebra representation of its Lie algebra $\mathfrak{g}$ on the same vector space V.

2 On the relation between Lie algebras and Lie groups

This section is meant as a complement of [2, Section 1.3.D] and contains some of the fundamental results clarifying the relationship between Lie algebras and Lie groups. Apart from Ado's theorem whose proof is not usually presented in standard textbooks, most of the results presented are proved in Lee's book [3].

If $\Phi : G \to H$ is a Lie group isomorphism, then as a consequence of Proposition [2, Proposition 1.13], we conclude that $\phi = T_{e_G}\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism. In particular we conclude that isomorphic Lie groups have isomorphic Lie algebras.

A natural question is the following: suppose that G and H are Lie groups for which we know that the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic. Under which conditions, if any, can we guarantee that G and H are isomorphic as Lie groups?

The following examples indicate some aspects that should be taken into account when attempting to give an answer at our question.

Example 1. Consider the Lie groups $O(2)$ and $SO(2)$. Both of them have the *same* (abelian) Lie algebra:

$$\mathfrak{o}(2) = \mathfrak{so}(2) = \operatorname{skew}(2) = \left\{ \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} : \omega \in \mathbb{R} \right\}.$$

However, it is clear that $O(2)$ and $SO(2)$ are *not* isomorphic. Indeed, they have a couple of fundamental differences:

(i) At the group level $SO(2)$ is abelian whereas $O(2)$ is non-abelian.

(ii) At the manifold level, $SO(2)$ is connected whereas $O(2)$ is not. (Recall that $SO(2)$ is defined as the connected component of $\mathbb{I}$ in $O(2)$.)

Example 1 in fact shows that a non-abelian, non-connected Lie group may have an abelian Lie algebra. This is in fact the only obstruction and the following theorem holds that generalises [2, Proposition 1.9].

Theorem 2.1. *If G is a connected Lie group, then G is abelian if and only if its Lie algebra is abelian.*

It turns out that connectedness of the Lie groups G and H is not sufficient a sufficient condition to guarantee that they are isomorphic Lie groups knowing that their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic Lie algebras as the following examples show.

Example 2. Consider the connected Lie groups $\mathbb{T}^n$ and $(\mathbb{R}^n, +)$. Both of them are abelian Lie groups so their Lie algebra coincides and equals $\mathbb{R}^n$ equipped with the zero bracket. It is clear that $\mathbb{T}^n$ and $(\mathbb{R}^n, +)$ are non-isomorphic Lie groups. One of them is compact and the other is not! Another topological difference between them is that $(\mathbb{R}^n, +)$ is simply connected whereas $\mathbb{T}^n$ is not.

Example 3. Consider the connected Lie groups $SO(3)$ and $SU(2)$. Their Lie algebras are

$$\mathfrak{so}(3) = \operatorname{skew}(3) = \{\xi \in L(3,\mathbb{R}) : \xi + \xi^T = 0\}$$
$$\mathfrak{su}(2) = \{\xi \in L(2,\mathbb{C}) : \xi + \xi^* = 0 \text{ and } \operatorname{Tr}(\xi) = 0\}.$$

A basis for $\mathfrak{so}(3)$ is given by $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ with

$$\hat{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As may be easily checked, these basis elements satisfy the commutation relations

$$[\hat{e}_1, \hat{e}_2] = \hat{e}_3, \quad [\hat{e}_3, \hat{e}_1] = \hat{e}_2, \quad [\hat{e}_2, \hat{e}_3] = \hat{e}_1.$$

On the other hand, a basis for $\mathfrak{su}(2)$ is given by $\{f_1, f_2, f_3\}$ with

$$f_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad f_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

As may be easily checked, these basis elements satisfy the commutation relations

$$[f_1, f_2] = f_3, \quad [f_3, f_1] = f_2, \quad [f_2, f_3] = f_1.$$

Based on the above, it is clear that the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic. Indeed, the linear function which takes the basis element $\hat{e}_i$ into f_i, $i = 1, 2, 3$, is a Lie algebra isomorphism.

However, the Lie groups $SO(3)$ and $SU(2)$ are *not* isomorphic! At the topological level $SO(3)$ is simply connected whereas $SU(2)$ is not.

Examples 2 and 3 show that connectedness is not sufficient to guarantee that Lie groups with isomorphic Lie algebras are isomorphic, and that simple-connectedness may play a role. In fact this is the only source of problems and the following theorem holds (compare with [2, Proposition 1.9]).

Theorem 2.2. *Let G and H be Lie groups and suppose that G is connected and simply connected. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\phi = T_{e_G}\Phi$.*

The content of the theorem can be illustrated by recalling that the maps:

$$\begin{aligned} &\Phi : (\mathbb{R}^n, +) \to \mathbb{T}^n, \qquad x \mapsto x \bmod 1 \\ &\mathcal{E} : S^3 \cong SU(2) \to \mathrm{SO}(3) \quad \text{(see [2, Proposition 1.13])} \end{aligned}$$

are Lie group homomorphisms which are local diffeomorphisms. As a consequence of [2, Proposition 1.9] their derivatives at the identity $T_0\Phi$, $T_e\mathcal{E}$, are Lie algebra isomorphisms.

Theorem 2.2 is the starting point to establish:

Corollary 2.3. *Let G and H be connected and simply connected Lie groups. Then G and H are isomorphic as Lie groups if and only if $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic as Lie algebras.*

An interesting extension of this result is [2, Proposition 1.23].

Another important, separate question in the theory of Lie groups is the following. We know ([2, Proposition 1.14]) that if G is Lie group and H is a Lie subgroup of G, then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$. It is natural to ask if *every* a Lie subalgebra of $\mathfrak{g}$ is the Lie algebra of a certain Lie subgroup of G. The answer to this question is provided by the following theorem.[1]

Theorem 2.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra then there exists a unique connected Lie subgroup $H \subset G$ such that $\mathfrak{h} = Lie(H)$. Moreover, $H = \langle \exp_G(\mathfrak{h}) \rangle$ meaning that the elements of H are of the form*

$$\exp_G(\xi_1)\exp_G(\xi_2)\cdots\exp_G(\xi_n)$$

for certain $\xi_1, \ldots, \xi_n \in \mathfrak{h}$.

As an illustration of the previous theorem consider $G = GL(2,\mathbb{R})$ whose Lie algebra is $\mathfrak{g} = L(2,\mathbb{R})$. It is a simple exercise to show that

$$\mathfrak{h} := \{\xi \in L(2,\mathbb{R}) : \mathrm{Tr}(\xi) = 0\},$$

is a subalgebra of $\mathfrak{g}$. The connected Lie subgroup H of $GL(2,\mathbb{R})$ whose existence is guaranteed by the theorem is $H = SL(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}) : \det(A) = 1\}$. Now, we know that the matrix exponential (which coincides with the Lie group exponential) $\mathrm{e} : \mathfrak{h} \to SL(2,\mathbb{R})$ is not surjective. For instance the matrix

$$\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

[1] The statement that $H = \langle \exp_G(\mathfrak{h}) \rangle$ is missing in Lee [3]. For this aspect see [1].

has no real logarithm. So it is not true that $SL(2,\mathbb{R})$ coincides with $e^{\mathfrak{h}}$. However, all elements of $SL(2,\mathbb{R})$ can be expressed as products of elements of $e^{\mathfrak{h}}$. For instance

$$\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \exp\begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}\exp\begin{pmatrix} \ln 2 & 0 \\ 0 & -\ln 2 \end{pmatrix}.$$

An important and deep result in the theory of Lie algebras, which relies only on their algebraic structure is Ado's Theorem that we state next.

Theorem 2.5 (Ado's Theorem)**.** *Let $(A,[\cdot,\cdot])$ be a finite dimensional real Lie algebra. Then $(A,[\cdot,\cdot])$ is isomorphic to a subalgebra of $L(n,\mathbb{R})$ for some $n \in \mathbb{N}$.*

Combining Theorems 2.4 and 2.5 we obtain:

Corollary 2.6. *Let $(A,[\cdot,\cdot])$ be a finite dimensional real Lie algebra. There exists a connected Lie subgroup G of $GL(n,\mathbb{R})$ (for some $n \in \mathbb{N}$) whose Lie algebra $\mathfrak{g}$ is isomorphic to $(A,[\cdot,\cdot])$.*

3 Relative equilibria

For simplicity, the discussion below is restricted to free and proper actions, although many of the concepts and results are valid without the freeness hypothesis under mild changes. So for the rest of this section we assume that

$$\Psi : G \times M \to M,$$

is a free and proper action of the group G on the manifold M.

3.1 Definition of relative equilibria and their angular velocity

Definition 3.1. Let $X \in \mathfrak{X}(M)$ be a G-invariant vector field. A ***relative equilibrium*** (RE) is an integral curve of X contained in a group orbit.

A RE is hence of the form $\Phi_t^X(m_0)$ for some initial condition $m_0 \in M$ and satisfies $\Phi_t^X(m_0) \in \mathcal{O}_{m_0}$ for all $t \in \mathbb{R}$. With a slight abuse of language, we will often refer to the initial condition $m_0 \in M$ as the RE. This simply means that the integral curve of X with initial condition m_0 is entirely contained in the group orbit through m_0.

If $\pi : M \to M/G$ is the orbit map, then the condition that $m_0 \in M$ is a RE is equivalent to

$$\pi(\Phi_t^X(m_0)) = \pi(m_0), \qquad \forall t \in \mathbb{R}. \tag{3.1}$$

Recall that under our hypothesis that the action is free and proper the quotient space M/G is a smooth manifold and $\pi : M \to M/G$ is a submersion ([2, Proposition 2.5]). As a consequence (see [2, Proposition 2.2]) there exists a reduced vector field $\bar{X} \in M/G$ which is π-related to X. Therefore, $\pi \circ \Phi_t^X = \Phi_t^{\bar{X}} \circ \pi$ and (3.1) may be rewritten as

$$\Phi_t^{\bar{X}}(\pi(m_0)) = \pi(m_0), \qquad \forall t \in \mathbb{R},$$

which means that $\pi(m_0) \in M/G$ is an equilibrium point of the reduced vector field $\bar{X}$. Therefore, relative equilibria correspond to equilibrium points of the reduced dynamics, which explains the terminology.

Proposition 3.2. *Let $m_0 \in M$ be a RE of the invariant vector field $X \in \mathfrak{X}(M)$. There exists a unique $\xi \in \mathfrak{g}$ such that*

$$X(m_0) = \xi_M(m_0).$$

Proof. First note that since $\Phi_t^X(m_0)$ is an integral curve of X then

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^X(m_0) = X(m_0).$$

On the other hand, since m_0 is a RE then $\Phi_t^X(m_0)$ is a curve contained in the group orbit $\mathcal{O}_{m_0}$. The assumption that the G-action is free and proper implies (see [2, Exercises 2.2.2(iii)]) that $\mathcal{O}_{m_0}$ is an embedded submanifold of M and therefore

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^X(m_0) \in T_{m_0}\mathcal{O}_{m_0}.$$

But on the other hand, by [2, Exercises 2.1.5(i)] we have

$$T_{m_0}\mathcal{O}_{m_0} = \{\eta_M(m_0) : \eta \in \mathfrak{g}\}.$$

This proves the existence of the element $\xi \in \mathfrak{g}$ in the statement of the proposition. To show uniqueness, suppose that $\xi_1, \xi_2 \in \mathfrak{g}$ satisfy $\left.\frac{d}{dt}\right|_{t=0} \Phi_t^X(m_0) = (\xi_i)_M(m_0)$, $i = 1, 2$, then

$$0 = (\xi_1)_M(m_0) - (\xi_2)_M(m_0) = (\xi_1 - \xi_2)_M(m_0).$$

Since the G-action is free, [2, Proposition 2.8] implies that $\xi_1 = \xi_2$. □

We will refer to the Lie algebra element $\xi \in \mathfrak{g}$ in the above proposition as the ***angular velocity*** of the RE $m_0 \in M$.

Remark 3.3. Note that m_0 is an equilibrium point of the invariant vector field $X \in \mathfrak{X}(M)$ if and only if m_0 is a RE with angular velocity $\xi = 0$.

Proposition 3.4. *Suppose that $m_0 \in M$ is a RE of $X \in \mathfrak{X}(M)$ with angular velocity $\xi \in \mathfrak{g}$. Then, for any $g \in G$, $\Psi_g(m_0) \in M$ is a RE of $X \in \mathfrak{X}(M)$ with angular velocity* $\mathrm{Ad}_g\xi \in \mathfrak{g}$.

Proof. The assumption that m_0 is a RE implies that $\Phi_t^X(m_0) \in \mathcal{O}_{m_0}$ for all $t \in \mathbb{R}$. On the other hand, by G-invariance of X, we have

$$\Phi_t^X(\Psi_g(m_0)) = \Psi_g(\Phi_t^X(m_0)), \qquad \forall t \in \mathbb{R},$$

which implies that $\Phi_t^X(\Psi_g(m_0)) \in \mathcal{O}_{m_0} = \mathcal{O}_{\Psi_g(m_0)}$ for all $t \in \mathbb{R}$. Therefore, $\Psi_g(m_0)$ is a RE.

To prove that the angular velocity of $\Psi_g(m_0)$ equals $\mathrm{Ad}_g\xi$ we use the orbit map $\Psi^m : G \to M$, $g \mapsto \Psi_g(m)$, for $m \in M$ and $g \in G$ and the following three identities

$$\eta_M(m) = T_e\Psi^m\eta, \qquad \forall m \in M,\ \eta \in \mathfrak{g}, \tag{3.2}$$

$$\Psi_g \circ \Psi^{m_0} = \Psi^{m_0} \circ L_g, \qquad \forall g \in G, \tag{3.3}$$

$$\Psi^{m_0} = \Psi^{\Psi_g(m_0)} \circ R_{g^{-1}}, \qquad \forall g \in G, \tag{3.4}$$

where in the last one we have abbreviated $g \cdot m := \Psi_g(m)$. The first identity follows from the definition of the infinitesimal generator $\eta_M \in \mathfrak{X}(M)$, and the other two are simple consequences of the definition of the orbit map and the action properties of Ψ.

Now, by G-invariance of X we have

$$X(\Psi_g(m_0)) = T_{m_0}\Psi_g(X(m_0)).$$

Using $X(m_0) = \xi_M(m_0) = T_e\Psi^{m_0}(\xi)$ we get

$$\begin{aligned}
X(\Psi_g(m_0)) &= T_{m_0}\Psi_g \circ T_e\Psi^{m_0}(\xi) \\
&= T_e(\Psi_g \circ \Psi^{m_0})(\xi) \quad \text{(chain rule)} \\
&= T_e(\Psi^{m_0} \circ L_g)(\xi) \quad \text{(by (3.3))} \\
&= T_e(\Psi^{\Psi_g(m_0)} \circ R_{g^{-1}} \circ L_g)(\xi) \quad \text{(by (3.4))} \\
&= T_e\Psi^{\Psi_g(m_0)} \circ T_e(L_g \circ R_{g^{-1}})(\xi) \quad \text{(chain rule)} \\
&= T_e\Psi^{\Psi_g(m_0)}(\mathrm{Ad}_g\xi) \quad \text{(definition of } \mathrm{Ad}_g) \\
&= (\mathrm{Ad}_g\xi)_M(\Psi_g(m_0)) \quad \text{(by (3.2))}.
\end{aligned}$$

□

Remark 3.5. Forgetting about the statement about the angular velocity, the above proposition shows that if $m_0 \in M$ is a RE, then $\Psi_g(m_0)$ is also a RE for all $g \in G$. This means that the whole orbit $\mathcal{O}_{m_0}$ consists of RE. In fact, some authors (e.g. Fassò [2]) define RE as group orbits which project to equilibrium points of the reduced dynamics.

Our goal in the following section will be to understand the dynamics of RE (under the additional assumption that the group G is compact). The following proposition is fundamental for this task and it is also useful in applications to determine existence of RE.

Proposition 3.6. *Let $\Psi : G \times M \to M$ be a free and proper Lie group action and $X \in \mathfrak{X}(M)$ be an invariant vector field. Let $m_0 \in M$ and $\xi \in \mathfrak{g}$. The following are equivalent.*

(i) *m_0 is a relative equilibrium with angular velocity ξ,*

(ii) *$X(m_0) = \xi_M(m_0)$,*

(iii) *$\Phi^X_t(m_0) = \Psi_{\exp(\xi t)}(m_0)$ for all $t \in \mathbb{R}$.*

Proof. $(i) \implies (ii)$ is just the definition of angular velocity (which recall is well-defined in virtue of Proposition 3.2).

$(ii) \implies (iii)$. By invariance of X and our hypothesis we have

$$\begin{aligned}
X(\Psi_{\exp(\xi t)}(m_0)) &= T_{m_0}\Psi_{\exp(\xi t)}X(m_0) \\
&= T_{m_0}\Psi_{\exp(\xi t)}\xi_M(m_0).
\end{aligned}$$

Now recall that $\Psi_{\exp(\xi t)}$ equals the flow $\Phi^{\xi_M}_t$ of the infinitesimal generator ξ_M. As first consequence of this, since $\xi_M = (\Phi^{\xi_M}_t)^*\xi_M = (\Psi_{\exp(\xi t)})^*\xi_M$, we conclude that

$$T_{m_0}\Psi_{\exp(\xi t)}\xi_M(m_0) = \xi_M(\Psi_{\exp(\xi t)}(m_0)).$$

As a second consequence, by definition of the flow, we have

$$\frac{d}{dt}\Psi_{\exp(\xi t)}(m_0) = \xi_M(\Psi_{\exp(\xi t)}(m_0)).$$

Combining the last identities yields

$$\frac{d}{dt}\Psi_{\exp(\xi t)}(m_0) = X(\Psi_{\exp(\xi t)}(m_0)),$$

which by the uniqueness of solutions to ODEs implies that $\Phi_t^X(m_0) = \Psi_{\exp(\xi t)}(m_0)$ for all t as required.

$(iii) \implies (i)$. Suppose that $\Phi_t^X(m_0) = \Psi_{\exp(\xi t)}(m_0)$ for all t. Then it is obviuos that $\Phi_t^X(m_0)$ is contained in the group orbit through m_0 and is therefore a RE. Its angular velocity is clearly ξ since $X(m_0) = \frac{d}{dt}\big|_{t=0}\Phi_t^X(m_0) = \frac{d}{dt}\big|_{t=0}\Psi_{\exp(\xi t)}(m_0) = \xi_M(m_0)$. □

The above proposition implies that for a RE $m_0 \in M$ with angular velocity $\xi \in \mathfrak{g}$ it holds

$$\Phi_t^X(m_0) = \Psi^{m_0}(\exp(\xi t)),$$

where, as usual, $\Psi^{m_0} : G \to M$ denotes the orbit map $g \mapsto \Psi_g(m_0)$. This suggests that in order to understand the dynamics of RE it is convenient to understand the properties of the one-parameter subgroup $\{\exp(t\xi)\}_{t\in\mathbb{R}}$, or, equivalently, the dynamics of the left invariant vector field $X_\xi \in \mathfrak{X}(G)$ which is what we will consider in the next section under the assumption that G is compact.

Examples

(i) Let $M = \mathbb{R}^2$ and consider the vector field $X(x,y) = (f(x), g(x))$ where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth. One can show that X is invariant under the free and proper action of $G = (\mathbb{R}, +)$ on $\mathbb{R}^2$ defined by

$$\Psi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2, \qquad \Psi(\lambda,(x,y)) = \Psi_\lambda(x,y) = (x, y+\lambda).$$

The infinitesimal generator of an element $\xi \in \mathbb{R} = \mathrm{Lie}(\mathbb{R})$ is the constant vector field $\xi_{\mathbb{R}^2}(x,y) = (0,\xi)$. The orbits are the vertical lines on the plane, so the orbit space $\mathbb{R}^2/\mathbb{R}$ is identified with $\mathbb{R}$ and the orbit projection map is $\pi : \mathbb{R}^2 \to \mathbb{R}$ is $\pi(x,y) = x$.

The reduced vector field is $\bar{X}(x) = f(x)$ and it can be checked to be π-related to X. The equilibrium points of $\bar{X}$ corresponds to the zeros of f so we conclude that the RE of X are points $(x_0, y) \in \mathbb{R}^2$ with the property that $f(x_0) = 0$. The angular velocity of $(x_0,y) \in \mathbb{R}^2$ is $\xi = g(x_0) \in \mathbb{R} = \mathrm{Lie}(\mathbb{R})$. In accordance with Proposition 3.6 we have

$$\Phi_t^X(x_0,y) = (x_0, y + g(x_0)t) = \Psi_{g(x_0)t}(x_0,y) = \Psi_{\exp_{\mathbb{R}}(g(x_0)t)}(x_0,y).$$

(ii) Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and consider the vector field $Y(x,y) = (-y,x)$. One can show that Y is invariant under the free[2] and proper action of $G = SO(2)$ on $\mathbb{R}^2$ defined by

$$\Psi : \mathrm{SO}(2)\times\mathbb{R}^2 \to \mathbb{R}^2, \qquad \Psi(R_\alpha,(x,y)) = \Psi_{R_\alpha}(x,y) = R_\alpha(x,y)^T, \qquad R_\alpha = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$$

The infinitesimal generator of an element $\xi \in \mathfrak{so}(2) = \mathrm{Lie}(SO(2))$ is the vector field $\xi_{\mathbb{R}^2}(x,y) = \xi(x,y)^T$. In other words

$$\text{if } \xi = \begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}, \quad \text{then} \quad \xi_{\mathbb{R}^2} = (-\omega y, \omega x).$$

[2]note that the action is free because we have removed the origin from $\mathbb{R}^2$.

The orbits are the circles centred at the origin, so the orbit space $M/SO(2)$ is identified with $\mathbb{R}^+ = (0,\infty)$ and the orbit projection map is $\pi : \mathbb{R}^2 \to \mathbb{R}^+$ is $\pi(x,y) = r = \sqrt{x^2+y^2}$.

The reduced vector field is $\bar{Y}(r) = 0$ and it can be checked (do it!) to be π-related to Y. The reduced vector field $\bar{Y}$ consists entirely of equilibrium points so we conclude that all $(x_0,y_0) \in \mathbb{R}^2$ are RE with angular velocity $\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In accordance with Proposition 3.6 we have

$$\Phi_t^Y(x_0,y_0) = R_t(x_0,y_0)^T = \Psi_{R_t}(x_0,y_0) = \Psi_{\exp(\xi t)}(x_0,y_0).$$

(iii) Consider the same phase space manifold $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and group action by $G = SO(2)$ as above, but consider now the vector field

$$Z = (-\lambda y + x(1-x^2-y^2), \lambda x + y(1-x^2-y^2))$$

It can be shown (do it!) that Z is invariant and the reduced vector field is $\bar{Z}(r) = r(1-r^2)$ and is π-related to Z. The equilibrium points of $\bar{Z}$ occur at $r=1$ so we conclude that all points in the unit circle are RE, for instance $(x_0,y_0) = (1,0)$. We have $Z(1,0) = (0,\lambda)$ which equals $\xi^2_{\mathbb{R}}(1,0)$ for

$$\xi = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}.$$

In accordance with Proposition 3.6 we have

$$\Phi_t^Z(1,0) = (\cos\lambda t, \sin\lambda t) = R_{\lambda t}(1,0)^T = \Psi_{R_{\lambda t}}(1,0) = \Psi_{\exp(\xi t)}(1,0).$$

3.2 The dynamics of relative equilibria of compact Lie groups

From now on assume that the group G is compact and (as before) suppose that $\Psi : G \times M \to M$ is a free and proper action[3], and suppose that $X \in \mathfrak{X}(M)$ is a G-invariant vector field. Our objective is to describe the dynamics of the RE of X.

3.2.1 Preliminary facts

Definition 3.7. A compact, connected, abelian subgroup of a compact Lie group G is called a ***torus subgroup*** of G.

The terminology is justified by the fact that any compact, connected, abelian group T is isomorphic as a Lie group to $\mathbb{T}^k$, where $k = \dim T$ (see [2, Proposition 2.6]). In particular, note that torus subgroups of a Lie group G are always embedded submanifolds diffeomorphic to $\mathbb{T}^k$ (torus subgroups are embedded submanifolds of G since they are closed in G (see [2, Proposition 1.9])).

For $0 \neq \xi \in \mathfrak{g}$ define

$$T_\xi := \overline{\{\exp(t\xi) : t \in \mathbb{R}\}} \subset G,$$

where, as usual, $\overline{A}$ denotes the topological closure of the set A.

[3] the properness is automatic since G is compact

Proposition 3.8. *For any $0 \neq \xi \in \mathfrak{g}$, T_ξ is a torus subgroup of G of dimension $k \geq 1$. (Therefore, T_ξ is isomorphic as a Lie group to $\mathbb{T}^k$.)*

Proof. By definition, T_ξ is a closed subset of the compact manifold G and therefore T_ξ is compact. Moreover, T_ξ is connected since it is obtained as the topological closure of the (path) connected set $\{\exp(t\xi)\}_{t\in\mathbb{R}}$.

Let us now check that T_ξ is an abelian subgroup of T_ξ. Suppose that $h \in T_\xi$. Then $h = \lim_{n\to\infty} \exp(t_n\xi)$ for a certain real sequence $\{t_n\}_{n\in\mathbb{N}}$. By continuity of the inversion map $i : G \to G$, $g \mapsto g^{-1}$, we have

$$h^{-1} = i(h) = i(\lim_{n\to\infty} \exp(t_n\xi)) = \lim_{n\to\infty} i(\exp(t_n\xi)) = \lim_{n\to\infty} \exp(-t_n\xi),$$

which shows that $h^{-1} \in T_\xi$. Now suppose that $k = \lim_{n\to\infty} \exp(s_n\xi) \in T_\xi$ for the real sequence $\{s_n\}_{n\in\mathbb{N}}$. By continuity of the product in G we have

$$\begin{aligned} hk &= (\lim_{n\to\infty} \exp(t_n\xi))(\lim_{n\to\infty} \exp(s_n\xi)) = \lim_{n\to\infty} \exp(t_n\xi)\exp(s_n\xi) = \lim_{n\to\infty} \exp((t_n+s_n)\xi) \\ &= \lim_{n\to\infty} \exp(s_n\xi)\exp(t_n\xi) = (\lim_{n\to\infty} \exp(s_n\xi))(\lim_{n\to\infty} \exp(t_n\xi)) = kh. \end{aligned}$$

The above calculation shows that $hk \in T_\xi$ and that $hk = kh$ which completes the proof that T_ξ is an abelian subgroup of G and, hence, since it is closed it is an embedded Lie subgroup of G (see [2, Proposition 1.9]). Note that the dimension of T_ξ is ≥ 1 since T_ξ contains the one-parameter subgroup $\{\exp(t\xi)\}_{t\in\mathbb{R}}$.

That T_ξ is isomorphic to $\mathbb{T}^k$ follows from [2, Proposition 2.6]. □

In the theory of compact Lie groups, a very important role is played by the so-called ***maximal tori*** which are torus subgroups of G that are not properly contained in any other torus subgroup (i.e. a torus subgroup T is a maximal torus if for any torus subgroup T' such that $T \subset T'$ one has $T' = T$). It is a fact that all maximal tori of a compact group G have the same dimension which is called the ***rank*** of G. In particular we conclude that if $0 \neq \xi \in \mathfrak{g}$, then $1 \leq \dim T_\xi \leq \operatorname{rank}(G)$.

Examples

(i) If $\xi = (\omega_1, \omega_2) \in \mathbb{R}^2 = \mathrm{Lie}(\mathbb{T}^2)$, then $\exp_{\mathbb{T}^2}(\xi t) = (\omega_1 t, \omega_2 t) \mod 1$. If $\omega_1/\omega_2 \notin \mathbb{Q}$ then $T_\xi = \mathbb{T}^2$. On the other hand, if $\omega_1/\omega_2 \in \mathbb{Q}$ then T_ξ is a 1-dimensional torus subgroup of $\mathbb{T}^2$.

(ii) Consider the Lie group $\mathbb{T}^3$, and the following vectors in $\mathbb{R}^3 = \mathrm{Lie}(\mathbb{T}^3)$:

$$\xi_1 = (1,1,1), \qquad \xi_2 = (1, \sqrt{2}), 0), \qquad \xi_3 = (1, \sqrt{2}, \pi).$$

Then $T_{\xi_1} = \{(\alpha,\alpha,\alpha) \mod 1 : \alpha \in \mathbb{R}\}$ which is a 1-dimensional torus subgroup of $\mathbb{T}^3$. On the other hand $T_{\xi_2} = \{(\alpha_1,\alpha_2,0) \mod 1 : \alpha_1,\alpha_2 \in \mathbb{R}\}$ which is a 2-dimensional torus subgroup of $\mathbb{T}^3$. Finally, $T_{\xi_3} = \mathbb{T}^3$.

(iii) If $\xi \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ then $\{e^{\hat{\xi} t}\}_{t\in\mathbb{R}}$ is a one dimensional torus subgroup of $SO(3)$ (the rotations of axis $\xi \in \mathbb{R}^3$). (In fact $\operatorname{rank}(SO(3)) = 1$).

(iv) Consider $U(2) = \{A \in GL(2,\mathbb{C}) : AA^* = \mathbb{I}\}$. Recall that $\mathfrak{u}(2) = \{\xi \in L(2,\mathbb{C}) : \xi + \xi^* = 0\}$. Let

$$\xi = \begin{pmatrix} i\omega_1 & 0 \\ 0 & i\omega_2 \end{pmatrix}, \qquad \omega_1, \omega_2 \in \mathbb{R}.$$

Then

$$\exp(\xi t) = \mathrm{e}^{\xi t} = \begin{pmatrix} e^{i\omega_1 t} & 0 \\ 0 & e^{i\omega_2 t} \end{pmatrix}$$

If $\omega_1/\omega_2 \notin \mathbb{Q}$ then

$$T_\xi = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} : \theta_1, \theta_2 \in [0, 2\pi) \right\}$$

which is a 2-dimensional torus subgroup of $U(2)$. (In fact $\operatorname{rank}(U(2)) = 2$).

(v) Consider $SO(4) = \{A \in GL(4,\mathbb{R}) : AA^T = \mathbb{I}\}$. Recall that $\mathfrak{so}(4) = \{\xi \in L(4,\mathbb{R}) : \xi + \xi^T = 0\}$. Let

$$\xi = \begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega 2 \\ 0 & 0 & \omega 2 & 0 \end{pmatrix}, \qquad \omega_1, \omega_2 \in \mathbb{R}.$$

Then

$$\exp(\xi t) = \mathrm{e}^{\xi t} = \begin{pmatrix} \cos\omega_1 t & -\sin\omega_1 t & 0 & 0 \\ \sin\omega_1 t & \cos\omega_1 t & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & -\sin\omega_2 t \\ 0 & 0 & \sin\omega_2 t & \cos\omega_2 t \end{pmatrix}$$

If $\omega_1/\omega_2 \notin \mathbb{Q}$ then

$$T_\xi = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in \mathrm{SO}(2) \right\}$$

which is a 2-dimensional torus subgroup of $SO(4)$. (In fact $\operatorname{rank}(SO(4)) = 2$).

In what follows we denote elements in $\mathbb{T}^k$ as $\langle\alpha\rangle := \alpha \mod 1$, for $\alpha \in \mathbb{R}^k$.

Lemma 3.9. *Let $\xi \in \mathfrak{g}$.*

(i) T_ξ is invariant under the flow of the left invariant vector field $X_\xi \in \mathfrak{X}(G)$,

(ii) the restriction of the flow of X_ξ to T_ξ is conjugate to a linear flow

$$(t, \langle\alpha\rangle) \mapsto \langle\alpha + \omega t\rangle$$

on $\mathbb{T}^k$, $k = \dim T_\xi$, with a frequency vector $\omega \in \mathbb{R}^k$.

Proof. (i) Let $h \in T_\xi$. By left invariance of X_ξ we have $\Phi_t^{X_\xi}(h) = h\Phi_t^{X_\xi}(e) = h\exp(t\xi) \in T_\xi$ since both h and $\exp(t\xi) \in T_\xi$ and T_ξ is a subgroup.

(ii) We will use that $\exp_{T_\xi} = \exp_G|_{T_\xi}$ which is true since T_ξ is a Lie subgroup of G (see [2, Corollary 1.18]). We will also use that $\xi \in \operatorname{Lie}(T_\xi)$ which is clear since the one-parameter subgroup $\{\exp(t\xi)\}_{t\in\mathbb{R}} \subset T_\xi$.

Since T_ξ is a torus subgroup of G of dimension $k \geq 1$, there exists a Lie group isomorphism $\Theta : \mathbb{T}^k \to T_\xi$. Let $\theta = T_{\langle 0\rangle}\Theta : \mathbb{R}^k = \operatorname{Lie}(\mathbb{T}^k) \to \operatorname{Lie}(T_\xi) \subset \mathfrak{g}$ be the corresponding Lie algebra isomorphism

[2, Proposition 1.13]. By [2, Proposition 1.18] we have the commuting diagram

$$\begin{tikzcd} \mathbb{T}^k \arrow[rr, "\Theta"] & & T_\xi \\ \mathbb{R}^k = \mathrm{Lie}(\mathbb{T}^k) \arrow[u, "\exp_{\mathbb{T}^k}"] \arrow[rr, "\theta"'] & & \mathrm{Lie}(T_\xi) \subset \mathfrak{g} \arrow[u, "\exp_{T_\xi} = \exp_G|_{T_\xi}"'] \end{tikzcd}$$

Considering that θ is a linear isomorphism, there exists a unique $\omega \in \mathbb{R}^k = \mathrm{Lie}(\mathbb{T}^k)$ such that $\theta(\omega) = \xi$. Using that $\exp_{\mathbb{T}^k}(t\omega) = \langle t\omega \rangle$ and in view of the above diagram we have

$$\Theta(\langle t\omega \rangle) = \exp_G(t\xi).$$

Let $\langle \alpha \rangle \in \mathbb{T}^k$ and multiply the above equality on the left by $\Theta(\langle \alpha \rangle)$. Using that Θ is a Lie group homomorphism and the definition of the group operation on $\mathbb{T}^k$ gives $\Theta(\langle \alpha \rangle)\Theta(\langle t\omega \rangle) = \Theta(\langle \alpha \rangle + \langle t\omega \rangle) = \Theta(\langle \alpha + t\omega \rangle)$ so we may write

$$\Theta(\langle \alpha + t\omega \rangle) = \Theta(\langle \alpha \rangle) \exp_G(t\xi).$$

But note that, by left invariance of X_ξ, we have

$$\Phi_t^{X_\xi}(\Theta(\langle \alpha \rangle)) = \Theta(\langle \alpha \rangle)\Phi_t^{X_\xi}(e_G) = \Theta(\langle \alpha \rangle) \exp_G(t\xi).$$

Therefore, combining the above identities we get

$$\Theta(\langle \alpha + t\omega \rangle) = \Phi_t^{X_\xi} \circ \Theta(\langle \alpha \rangle),$$

which can be expressed as the commuting diagram

$$\begin{tikzcd} \mathbb{T}^k \arrow[r, "\Theta"] \arrow[d, "\Phi_t^{X_\omega^{\mathbb{T}^k}}"'] & T_\xi \arrow[d, "\Phi_t^{X_\xi}|_{T_\xi}"] \\ \mathbb{T}^k \arrow[r, "\Theta"'] & T_\xi \end{tikzcd} \tag{3.5}$$

where $\Phi_t^{X_\omega^{\mathbb{T}^k}}(\langle \alpha \rangle) = \langle \alpha + t\omega \rangle$. This shows that the restriction of the flow of X_ξ to T_ξ is conjugate to the linear flow on the torus $\mathbb{T}^k$ with frequency $\omega \in \mathbb{R}^k$. $\square$

We finish this section by considering $gT_\xi = \{gh : h \in T_\xi\}$. Given that $gT_\xi = L_g(T_\xi)$ and that $L_g : G \to G$ is a diffeomorphism and T_ξ is a submanifold, it follows[4] that gT_ξ is a submanifold of G diffeomorphic to T_ξ, and hence also to $\mathbb{T}^k$ where $k = \dim T_\xi$. (Note however that gT_ξ need not contain the identity element e_G for a general $g \in G$ and hence it need not be a Lie subgroup.)

3.2.2 Dynamics of relative equilibria

Throughout this section, we continue to assume that $\Psi : G \times M \to M$ is a free and proper action of the compact Lie group G and we let $X \in \mathfrak{X}(M)$ be an invariant vector field.

Let $m_0 \in M$ be a RE of X with angular velocity $0 \neq \xi \in \mathfrak{g}$ and suppose that $\dim T_\xi = k \geq 1$. For $g \in G$ we define

$$\Sigma_g := \Psi^{m_0}(gT_\xi) = \{\Psi_h(m_0) : h \in gT_\xi\}.$$

It is clear that $\Sigma_g \subset \mathcal{O}_{m_0}$ and that $\mathcal{O}_{m_0} = \bigcup_{g \in G} \Sigma_g$.

[4] the restriction of a diffeomorphism to a submanifold is always a diffeomorphism onto its image.

Proposition 3.10. *Let $\Psi : G \times M \to M$ be a free and proper action of the compact Lie group G and let $X \in \mathfrak{X}(M)$ be an invariant vector field. Suppose that $m_0 \in M$ is a RE of X with angular velocity $0 \neq \xi \in \mathfrak{g}$. For any $g \in G$ we have:*

(i) *Σ_g is an embedded submanifold of $\mathcal{O}_{m_0}$ diffeomorphic to $\mathbb{T}^k$ and invariant under the flow of X, where $k = \dim T_\xi$ satisfies $1 \leq k \leq \operatorname{rank}(G)$;*

(ii) *the restriction of the flow of X to Σ_g is conjugate to a linear flow*

$$(t, \langle \alpha \rangle) \mapsto \langle \alpha + \omega t \rangle$$

on $\mathbb{T}^k$, with a frequency vector $\omega \in \mathbb{R}^k$ that depends only on ξ (not on $g \in G$).

Proof. Recall that the orbit map $\Psi^{m_0} : G \to \mathcal{O}_{m_0}$ is a diffeomorphism and that gT_ξ is an embedded submanifold of G. It follows that the restriction $\Psi^{m_0}|_{gT_\xi} : gT_\xi \to \Sigma_g$ is a diffeomorphism. But gT_ξ is compact and diffeomorphic to $\mathbb{T}^k$ so $\Psi^{m_0}|_{gT_\xi} : gT_\xi \to \Sigma_g$ is actually an embedding[5] and Σ_g is an embedded submanifold diffeomorphic to $\mathbb{T}^k$.

To prove the remaining statements consider first the case in which $g = e$, the group identity. Let $m \in \Sigma_e = \Psi^{m_0}(T_\xi)$. Then $m = \Psi^{m_0}(h) = \Psi_h(m_0)$ for a unique $h \in T_\xi$. Let $t \in \mathbb{R}$. Using invariance of X we have

$$\Phi_t^X(m) = \Phi_t^X(\Psi_h(m_0)) = \Psi_h(\Phi_t^X(m_0)).$$

By item (iii) of Proposition 3.6 we have $\Phi_t^X(m_0) = \Psi_{\exp(t\xi)}(m_0)$ and therefore,

$$\Phi_t^X(m) = \Psi_h(\Psi_{\exp(t\xi)}(m_0)) = \Psi_{h\exp(t\xi)}(m_0) = \Psi^{m_0}(h\exp(t\xi)).$$

Since $\exp(t\xi), h \in T_\xi$ and T_ξ is a subgroup we conclude that $h\exp(t\xi) \in T_\xi$ and hence $\Phi_t^X(m) \in \Psi^{m_0}(T_\xi) = \Sigma_e$, which shows that Σ_e is invariant under the flow of X as required. Moreover, using that the flow of the left invariant vector field $X_\xi \in \mathfrak{X}(G)$ satisfies $\Phi_t^{X_\xi}(h) = h\exp(t\xi)$, the above identities imply:

$$\Phi_t^X \circ \Psi^{m_0}(h) = \Psi^{m_0} \circ \Phi_t^{X_\xi}(h), \qquad \forall h \in T_\xi,\ \forall t \in \mathbb{R},$$

so for all $t \in \mathbb{R}$ we have the commuting diagram

$$\begin{tikzcd} T_\xi \arrow[r, "\Psi^{m_0}|_{T_\xi}"] \arrow[d, "\Phi_t^{X_\xi}|_{T_\xi}"'] & \Sigma_e \arrow[d, "\Phi_t^X|_{\Sigma_e}"] \\ T_\xi \arrow[r, "\Psi^{m_0}|_{T_\xi}"'] & \Sigma_e \end{tikzcd} \tag{3.6}$$

In other words, the diffeomorphism $\Phi_t^{X_\xi}\big|_{T_\xi} : T_\xi \to \Sigma_e$ conjugates the restriction of the flow of X_ξ to T_ξ to the restriction of the flow of X to Σ_e.

Now note that for $g \in G$ we have $\Psi_g(\Sigma_e) = \Sigma_g$. Indeed,

$$\Psi_g(\Sigma_e) = \Psi_g(\Psi^{m_0}(T_\xi)) = \Psi_g(\Psi_{T_\xi}(m_0)) = \Psi_{gT_\xi}(m_0) = \Psi^{m_0}(gT_\xi) = \Sigma_g.$$

[5] since $\Psi^{m_0}|_{gT_\xi}$ is a diffeomorphism onto its image, it is an immersion. And any compact immersed submanifold is embedded.

Denote by $\psi_g := \Psi_g|_{\Sigma_e} : \Sigma_e \to \Sigma_g$. Considering that $\Psi_g : M \to M$ is a diffeomorphism and Σ_e is an embedded submanifold it follows that $\psi_g : \Sigma_e \to \Sigma_g$ is a diffeomorphism. Now, using left invariance of X, for $m \in \Sigma_e$ we have

$$\psi_g \circ \Phi_t^X(m) = \psi_g(\Phi_t^X(m)) = \Psi_g(\Phi_t^X(m)) = \Phi_t^X(\Psi_g(m)) = \Phi_t^X(\psi_g(m)) = \Phi_t^X \circ \psi_g(m).$$

Hence, the diffeomorphism $\psi_g : \Sigma_e \to \Sigma_g$ conjugates the restriction of the flow of X to Σ_e to the restriction of the flow of X to Σ_g and we can augment the commuting diagram (3.6) to

$$\begin{tikzcd} T_\xi \arrow[r, "\Psi^{m_0}|_{T_\xi}"] \arrow[d, "\Phi_t^{X_\xi}|_{T_\xi}"'] & \Sigma_e \arrow[r, "\psi_g"] \arrow[d, "\Phi_t^X|_{\Sigma_e}"] & \Sigma_g \arrow[d, "\Phi_t^X|_{\Sigma_g}"] \\ T_\xi \arrow[r, "\Psi^{m_0}|_{T_\xi}"'] & \Sigma_e \arrow[r, "\psi_g"] & \Sigma_g \end{tikzcd}$$

The proof is completed by noting that the horizontal maps in the diagram are diffeomorphisms and using Lemma 3.9 that guarantees that the flow of X_ξ restricted to T_ξ is conjugate to a linear flow on the torus $\mathbb{T}^k$ with frequency vector $\omega \in \mathbb{R}^k$ (i.e. complementing the above diagram with (3.5)). $\square$

The above proposition gives the following detailed description of the dynamics of the RE. The orbit $\mathcal{O}_{m_0}$ is comprised of the embedded submanifolds Σ_g, $g \in G$, each of which is diffeomorphic to a k-dimensional torus and is invariant under the flow of X. The dimension k of the these submanifolds is $k = \dim T_\xi$ and satisfies $1 \le k \le \mathrm{rank}(T_\xi)$. Moreover, the restriction of the flow of X to any of these invariant submanifolds is conjugate to a linear flow on the torus $\mathbb{T}^k$ with a frequency vector $\omega \in \mathbb{R}^k$ which is constant for all of them.

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