MATH4U - 1/12/2022

Thursday, 24 November 2022 22:52

1) QUESTIONS ABOUT PREVIOUS EXERCISES

2) STUDY OF A FUNCTION

$$\boxed{13)\quad f(x) = e^{3x}\left(x^2 - |x+1|\right)}$$

$D: \mathbb{R}$

$\operatorname{sgn} f(x)$: $e^{3x} > 0 \quad \forall x$

$x^2 - |x+1| \geq 0$ if $x+1 \geq 0$ $\quad x^2 - x - 1 \geq 0$ (A)

if $x+1 < 0$ $\quad x^2 + x + 1 \geq 0$ (B)

(A) $x_{12} = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

$\dfrac{1-\sqrt{5}}{2} > -1 \quad 1 - \sqrt{5} > -2$

$-\sqrt{5} > -3 \quad \sqrt{5} < 3$ ✓

$\left[-1, \dfrac{1-\sqrt{5}}{2}\right] \cup \left[\dfrac{1+\sqrt{5}}{2}, +\infty\right[$

(B) $x_{12} = \dfrac{-1 \pm \sqrt{1-4}}{2}$ $\times$ $\Delta < 0$ $\quad \left]-\infty, -1\right[$

$f(x) \geq 0 \quad x$ in $\left]-\infty, \dfrac{1-\sqrt{5}}{2}\right] \cup \left[\dfrac{1+\sqrt{5}}{2}, +\infty\right[$

$\dfrac{1-\sqrt{5}}{2} \to \sim -\dfrac{1}{2}$ $\quad \dfrac{1+\sqrt{5}}{2} \to \sim 3/2$

$f\left(\dfrac{1 \pm \sqrt{5}}{2}\right) = 0$

$\lim_{x \to +\infty} e^{3x}\left(x^2 - |x+1|\right) \sim \lim_{x \to +\infty} e^{3x} x^2 = +\infty$

$\lim_{x \to -\infty} e^{3x}\left(x^2 - |x+1|\right) \sim \lim_{x \to -\infty} e^{3x} x^2 = 0^+$ HORIZONTAL ASYMTOTE

continuity : $f(x)$ is continuous in $D = \mathbb{R}$

derivability :

$f'(x) = \dfrac{d}{dx}\left(e^{3x}\left(x^2 - (x+1)\operatorname{sgn}(x+1)\right)\right)$

$= 3e^{3x}\left(x^2 - (x+1)\operatorname{sgn}(x+1)\right) + e^{3x}\left(2x - \operatorname{sgn}(x+1)\right)$

$= e^{3x}\left(3x^2 + 2x - \operatorname{sgn}(x+1)(3x+4)\right)$

$$\overset{!}{=} e^{3x}\left(3x^2+2x-\text{sgn}(x+1)(3x+4)\right)$$

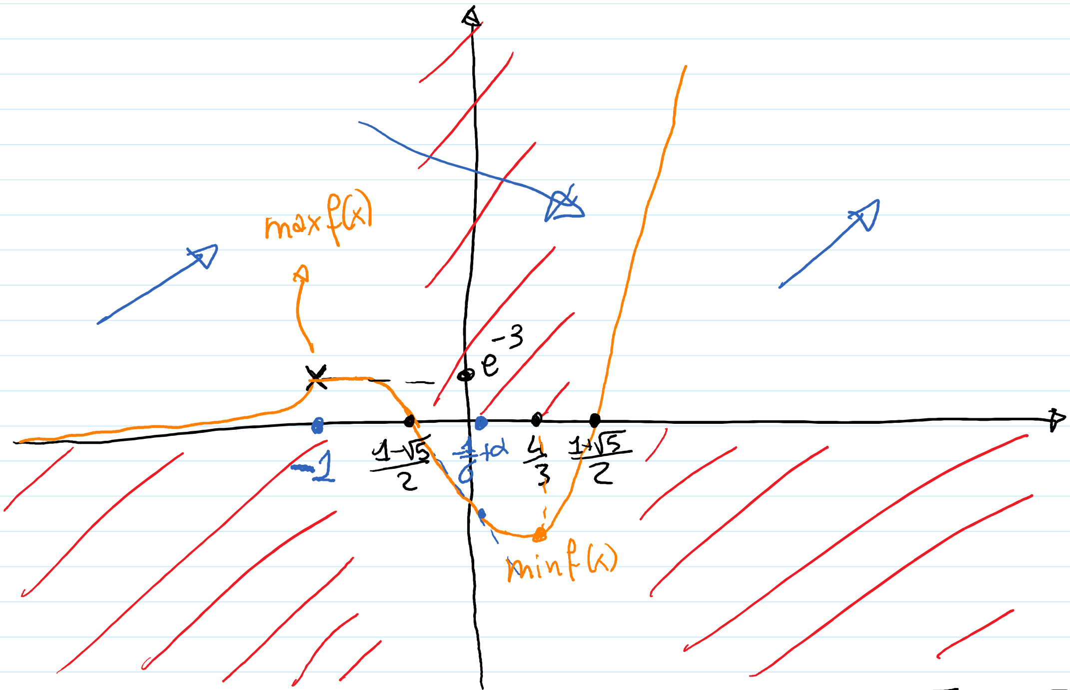
$$\overset{!}{=} e^{3x}\left(3x^2+(2-3\,\text{sgn}(x+1))x-4\,\text{sgn}(x+1)\right)$$

$$\lim_{x\to -1^-} e^{3x}\left(3x^2+(2+3)x+4\right) \overset{?}{=} \lim_{x\to -1^+} e^{3x}\left(3x^2+(2-3)x-4\right)$$

$$e^{-3}(3+5(-1)+4) \qquad\qquad e^{-3}(3+(-1)(-1)-4)$$

$$2e^{-3} \neq 0$$

$f(x)$ is not derivable in $x=-1$
$f(x)$ is derivable in $\mathbb{R}\setminus\{-1\}$



$$f(-1)=e^{-3}(1-0)=e^{-3}<1$$

$$\left\{\begin{array}{l}\frac{1+\sqrt5}{2}>\frac{4}{3}\\ 3+3\sqrt5>8\\ 3\sqrt5>5\\ 3>\sqrt5\end{array}\right\}$$

$$e^{3x}\left(3x^2+2x-\text{sgn}(x+1)(3x+4)\right)>0$$

$$\left(e^{3x}\left(3x^2+(2-3\,\text{sgn}(x+1))x-4\,\text{sgn}(x+1)\right)>0\right)$$

$$e^{3x}>0$$

$$\left(3x^2+2x-\text{sgn}(x+1)(3x+4)\right)>0$$

if $x+1>0$ $\quad 3x^2+2x-(3x+4)=3x^2-x-4>0$ $\quad$ (A)
if $x+1<0$ $\quad 3x^2+2x+(3x+4)=3x^2+5x+4>0$ $\quad$ (B)

(A) $3x^2-x-4>0 \qquad x_{12}=\dfrac{1\pm\sqrt{1+4\cdot4\cdot3}}{2\cdot 3}=\dfrac{1\pm7}{6}=-1,\ \dfrac{4}{3}$

f is increasing for x in $[\frac{4}{3},+\infty[$

(B) $3x^2+5x+4>0 \qquad x_{12}=-5\pm\sqrt{25-4\cdot4\cdot3}\ \times$

$\hookrightarrow \Delta<0$

f is increasing for x in $]-\infty;-1[$

$f(\frac{4}{3})=\frac{-5}{9}e^4$

$$f''(x)=\frac{d}{dx}\left(e^{3x}\left(3x^2+2x-\operatorname{sgn}(x+1)(3x+4)\right)\right)$$

$$=3e^{3x}(-\,-\,-\,-)+e^{3x}\left(6x+2-3\operatorname{sgn}(x+1)\right)$$

$$=e^{3x}\left(9x^2+12x-\operatorname{sgn}(x+1)(9x+15)+2\right)$$

$e^{3x}>0$

$(-\!-\!-)>0 \qquad$ if $x+1>0 \quad 9x^2+3x-13>0$

if $x+1<0 \quad 9x^2+21x+17>0 \qquad \Delta<0$

$$x_{12}=\frac{-3\pm\sqrt{9+4\cdot13\cdot9}}{18}=-\frac{1}{6}\pm\alpha \ \hookrightarrow \sim 1.5$$

$f(x)$ convex in $]-\frac{1}{6}+\alpha,+\infty[$
$\cup$
$]-\infty,-1[$

$$\boxed{f(x)=\lambda^2 x\, e^{\frac{1}{\lambda x}} \qquad \lambda\in\mathbb{R}}$$

if $\lambda=0$ $f(x)$ is not defined

$\mathbb{D} \qquad \lambda x\neq0 \rightarrow x\neq0 \qquad \mathbb{D}=\mathbb{R}\setminus\{0\} \qquad \forall\lambda\neq0$

$$f'(x)=\frac{d}{dx}\left(\lambda^2 x\, e^{\frac{1}{\lambda x}}\right)$$

$$=\lambda^2 e^{\frac{1}{\lambda x}}+\lambda^2 x\, e^{\frac{1}{\lambda x}}\cdot\left(\frac{-1}{(\lambda x)^2}\cdot\lambda\right)$$

$$= \lambda^2 e^{\frac{1}{\lambda x}} + \lambda^2 x e^{\frac{1}{\lambda x}} \cdot \left(\frac{-1}{(\lambda x)^2} \cdot \lambda \right)$$

$$= e^{\frac{1}{\lambda x}} \left(\lambda^2 + \frac{\cancel{\lambda^2} \cancel{x} (-\lambda)}{\cancel{\lambda^2} x^{\cancel{2}}} \right) = e^{\frac{1}{\lambda x}} \left(\lambda^2 - \frac{\lambda}{x} \right)$$

$f'(x) > 0$ $\quad e^{\frac{1}{\lambda x}} > 0 \quad \forall x \, \forall \lambda$

$\lambda \left(\lambda - \frac{1}{x} \right) > 0 \qquad \frac{\lambda(\lambda x - 1)}{x} > 0 \qquad$ N $x = 1/\lambda$ $\Rightarrow$ >0 $\lambda > 0$; <0 $\lambda < 0$

D $x = 0$

	$\lambda > 0$						$\lambda < 0$				
		0		$1/\lambda$				$1/\lambda$		0	
λ	$+$	0	$+$		$+$		$-$		$-$	0	$-$
$\lambda x - 1$	$-$		$-$	0	$+$		$+$	0	$-$		$-$
x	$-$	$\nexists$	$+$		$+$		$-$		$+$	$\nexists$	$+$
$g(x)$	$+$	$\nexists$	$-$	0	$+$		$+$	0	$-$	$\nexists$	$+$

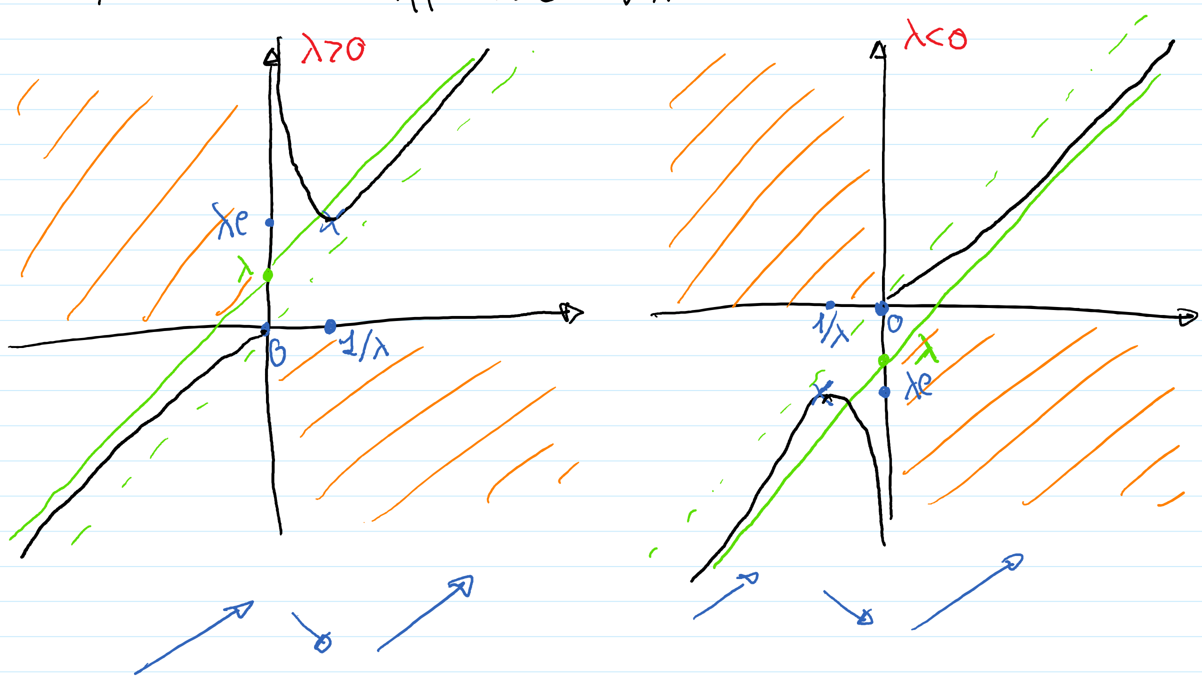
$\min f(x)$ $\qquad$ $\max f(x)$

$\lambda x - 1 > 0$

$\lambda x > 1$

$x < \frac{1}{\lambda}$

$f(x) > 0 \qquad \lambda^2 x e^{\frac{1}{\lambda x}} > 0 \quad$ iff $\quad x > 0 \quad \forall \lambda$



$$f\left(\frac{1}{\lambda}\right) = \lambda^2 \frac{1}{\lambda} e^{\frac{\lambda}{\lambda}} = \lambda e$$

LIMITS

$\lim \ldots \frac{1}{\ldots} \rightarrow \lambda > 0 \quad \lim \lambda^2 x \cdot 1 = +\infty$

LIMITS

$$\lim_{x\to+\infty} \lambda^2 x e^{\frac{1}{\lambda x}} \to \lambda>0 \quad \lim_{x\to+\infty} \lambda^2 x \cdot 1 = +\infty$$
$$\to \lambda<0 \quad \lim_{x\to+\infty} \lambda^2 x \cdot 1 = +\infty$$

$$\lim_{x\to-\infty} \lambda^2 x e^{\frac{1}{\lambda x}} \to \lambda\neq 0 \quad \lim_{x\to-\infty} \lambda^2 x = -\infty$$

$$\lim_{x\to 0^+} \lambda^2 x e^{\frac{1}{\lambda x}} \to \lambda>0 \quad \lim_{x\to 0^+} \lambda^2 x e^{+\infty} = +\infty$$
$$\to \lambda<0 \quad \lim_{x\to 0^+} \lambda^2 x e^{-\infty} = 0^+$$

$$\lim_{x\to 0^-} \lambda^2 x e^{\frac{1}{\lambda x}} \to \lambda>0 \quad \lim_{x\to 0^-} \lambda x e^{-\infty} = 0^-$$
$$\to \lambda<0 \quad \lim_{x\to 0^-} \lambda x e^{+\infty} = -\infty$$

ASYM ? $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \lambda^2 e^{\frac{1}{\lambda x}} = \lambda^2$

$$\lim_{x\to\pm\infty} f(x) - \lambda^2 x = \lim_{x\to\pm\infty} \lambda^2 x \left(\underbrace{e^{\frac{1}{\lambda x}}}_{1+\frac{1}{\lambda x}+o\left(\frac{1}{x}\right)} - 1\right) \approx \lim_{x\to\pm\infty} \lambda^{\cancel{2}} \cancel{x} \cdot \frac{1}{\cancel{\lambda x}} + o(1)$$
$$= \lambda$$

LINE : $y = \lambda^2 x + \lambda$

ADDITIONAL EXERCISES

11) $f(x) = \begin{cases} e^{-\frac{1}{|x|}} (2-3|x|) & x\neq 0 \\ 0 & x=0 \end{cases}$

14) $f(x) = e^{3x} |x^2-3|$

15) $f(x) = \operatorname{arctg}\left(\frac{2x+1-|x|}{|x+1|}\right)$

17) $f(x) = \log\left(e^{\frac{x}{2}} - \sqrt{|2-e^x|}\right)$ → Note: Desmos.com solution is imprecise due to limited numerical precision

21) $f(x) = 2x + \frac{\sinh|x|}{\sinh x - 1}$

You can find additional functions in the typed notes and in the collection of previous exams