Definition

Lebesgue
more advanced

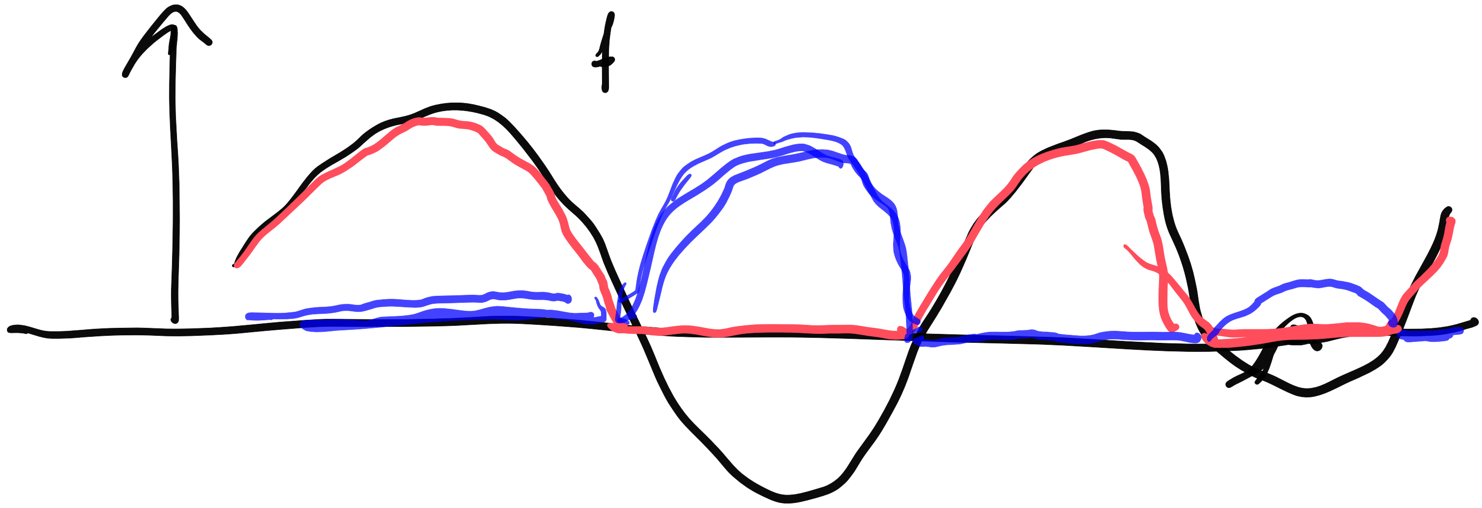
Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function.

We say that f is Riemann-integrable if $A(f^+)$ and $A(f^-)$ exist and we call

$$\int_a^b f(x)\,dx := A(f^+) - A(f^-)$$

the integral of f on $[a,b]$.

$$f \in \mathcal{R}([a,b])$$


f

Elementary properties of the integral:

1) linearity

$f : [a,b] \to \mathbb{R}$
$g : [a,b] \to \mathbb{R}$

$f, g \in \mathcal{R}([a,b])$, $\alpha, \beta \in \mathbb{R}$, then

$\alpha f + \beta g \in \mathcal{R}([a,b])$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$$

2) monotonicity

$f, g \in \mathcal{R}([a,b])$, if

$f(x) \leq g(x)$ $\forall x \in [a,b]$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

3) "triangular" inequality

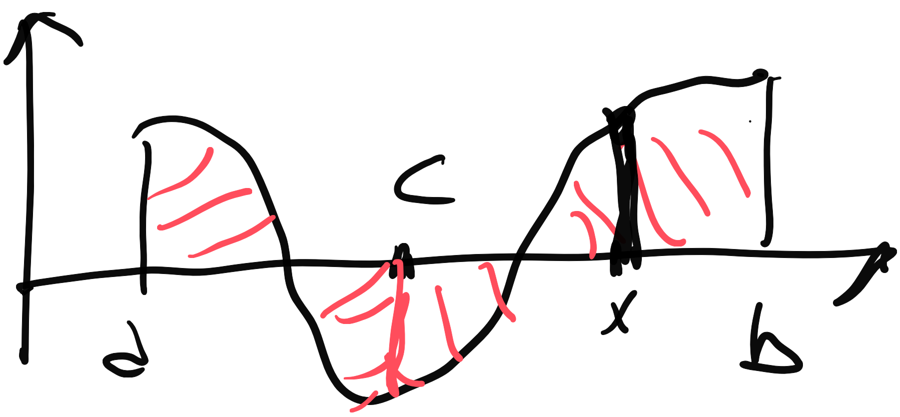
$f \in \mathcal{R}([a,b])$ then $|f| \in \mathcal{R}([a,b])$

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$$

4) decomposition

$a \le c \le b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Mean value theorem:

$f \in \mathcal{C}([a,b]) \quad (\Rightarrow \mathcal{R}([a,b]))$

Then there exists $\xi \in [a,b]$ such that

$$\frac{\int_a^b f(x)\,dx}{b-a} = f(\xi)$$

Proof: by Weierstrass

$\exists\ x_M$ maximum point

〃 x_m 〃 〃

$$m = f(x_m) \leq f(x) \leq f(x_M) = M$$

$$m \leq f(x) \leq M$$

$$\Downarrow$$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$m(b-a) \le \int_a^b f(x)\,dy \le M(b-a)$$

$$f(x_m) \le \boxed{\frac{\int_a^b f(x)\,dx}{b-a}} \le f(x_M)$$

$\Longrightarrow \exists\, \xi \in [a,b]$ such that

$$f(\xi) = \frac{\int_a^b f(x)\,dx}{b-a}$$

The "integral function"

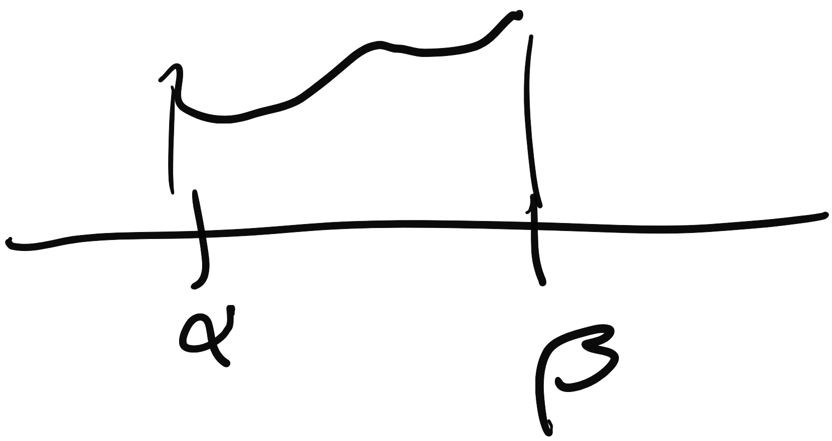
Begin with a definition:

$f : [a, b] \longrightarrow \mathbb{R}$
integrable

If $\alpha < \beta$ $\quad (\alpha, \beta \in [a, b])$

we set

$$\int_\beta^\alpha f(x)\,dx := -\int_\alpha^\beta f(x)\,dx$$

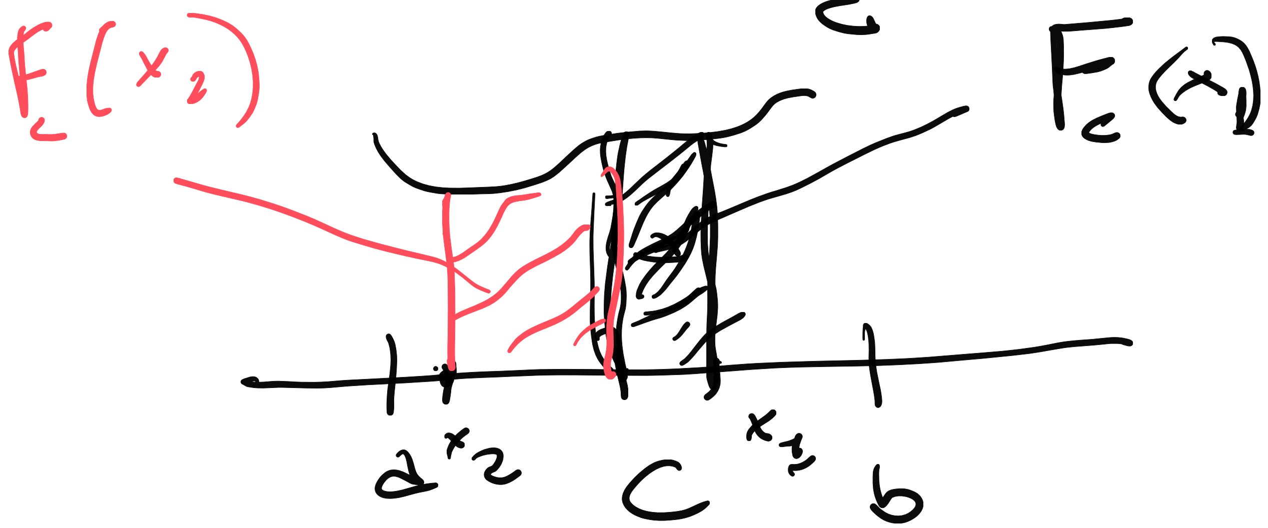


Now choose $c \in [a,b]$ and define the

integral function from c:

$$F_c : [a,b] \longrightarrow \mathbb{R}$$

$$F_c(x) = \int_c^x f(t)\,dt$$



$$F_c(c) = \int_c^c f(t)\,dt = 0$$

To state the main result we need a definition:

$f : [a,b] \longrightarrow \mathbb{R}$

bounded. We say that

$\Phi : [a,b] \longrightarrow \mathbb{R}$

is a primitive of f

if

$$\Phi'(x) = f(x)$$

For instance:

$f = \sin(x)$ $\qquad$ $F = -\cos x$ is a primitive.

$\tilde{F} = F + 1 = -\cos x + 1$

$F_{(k)} = F + k = -\cos x + k$

$k \in \mathbb{R}$

G is a primite of f

F " " " of f

$G'(x) - F'(x) = (G - F)'(x) =$

$f(x) - f(x) = 0$

$(G - F)' = 0$

because the domain is an interval

$\longrightarrow G(x) - F(x) = k \in \mathbb{R}$

$\cos(x)$ a primitive is $\sin(x)$

$$\int \cos x \, dx = \{ \sin(x) + k \quad k \in \mathbb{R} \}$$

set of primitives of $\cos x$

$$\int f(x)\, dx = \{ \text{primitives of } f \}$$

$$\boxed{\int \frac{1}{1+x^2} dx} = \operatorname{arctg}(x) + k$$

$$\left(\operatorname{arctg}(x) \right)' = \frac{1}{1+x^2}$$

$$\|$$

$$\frac{1}{(tg)'(\operatorname{arctg} x)} = \cos^2(\operatorname{arctg} x) = \frac{1}{\frac{1}{\cos^2(\operatorname{arctg})}}$$

$$\left(\frac{\sin^2(\operatorname{arctg} x)+\cos^2(\operatorname{arctg} x)}{\cos^2(\operatorname{arctg} x)}\right)^{-1} =$$

$$= \left(\operatorname{tg}^2(\operatorname{arctg}(x)) + 1\right)^{-1} =$$

$$\frac{1}{x^2+1}$$

In general it is not immediat to find a primitive.

$f(x) = x\cos x$

Derivare humanum est

Integrare diabolicum

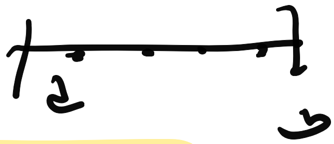
Problem: Given $f(x)$, does a primitive exist?

Theorem. Let

$$f : [a,b] \longrightarrow \mathbb{R}$$

be continuous.

For any $c \in [a,b]$



the integral function

$$x \longmapsto F_c(x) := \int_c^x f(t)\,dt$$

is differentiable and

$$F_c'(x) = f(x) \quad \forall x \in [a,b],$$

namely F_c is a primitive.

<u>Proof.</u> We want to show that $F_c'(x) = f(x)$ $\forall x \in [a,b]$

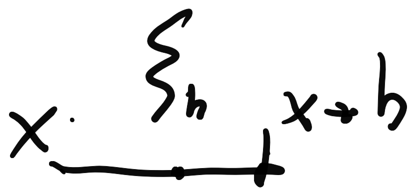
$$F'_{c+}(x) = \lim_{h\to 0} \frac{F_c(x+h) - F_c(x)}{h} =$$

$$\lim_{h\to 0+} \frac{\int_c^{x+h} f(t)\,dt - \int_c^{x} f(t)\,dt}{h} =$$

$$\lim_{h\to 0+} \frac{\cancel{\int_c^{x} f(t)\,dt} + \int_x^{x+h} f(t)\,dt - \cancel{\int_c^{x} f(t)\,dt}}{h} =$$

$$\lim_{h\to 0+} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0} f(\xi_h) =$$

mean value Th



$x \le \xi_n \le x+h$ if $h \to 0+$

by 2 police - - -

$\xi_n \to x+$

Since f is continuous

$$\lim_{h\to 0} f(\xi_n) = f(x)$$

$$= f(x)$$

$$F_c'(x) = \lim_{h\to 0-} \frac{F_c(x+h) - F_c(x)}{h}$$

. . . .

. . . . . $= f(x)$

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q.e.d.

Obsere: choose $\hat{c} \neq c$

$$F_{\hat{c}}'(x) = f(x) = F_c'(x)$$

$$F_{\hat{c}}(x) = \int_{\hat{c}}^{x} f(t)\,dt$$

$$F_c(x) = \int_{c}^{x} f(t)\,dt$$

$$F_{\hat{c}}(x) - F_c(x) = \int_{\hat{c}}^{x} f\,dt - \int_{c}^{x} f(t) =$$

$$= \int_{\hat{c}}^{x} f(t)\,dt + \int_{x}^{c} f(t)\,dt =$$

$$= \int_{\hat{c}}^{c} f(t)\,dt = \text{Constant}$$

$$f = x\cos x$$

$$F = x\sin x - \sin x$$

$$F' = \cancel{\sin x} + x\cos x - \cancel{\sin x}$$

$$H : [a,b] \to \mathbb{R}$$

$$K : [a,b] \to \mathbb{R}$$

differentiable

$$(HK)'_{(x)} = H'_{(x)}K_{(x)} + H_{(x)}K'_{(x)}$$

$$H'K = (HK)' - HK'$$

$$\int_c^x H'K = \int_c^x (HK)' - \int_c^x (HK')$$

Integration by parts

$$\int_a^b (f \cdot g')\,dx = f \cdot g\Big]_a^b - \int_a^b f' g\,dx$$

$$= f(b)g(b) - f(a)g(a) - \int_a^b \cdots$$

$$\int_a^x \underset{f}{t}\, \underset{g'}{\cos t}\, dt = t \sin t\Big|_a^x - \int_a^x \cos t\, dt$$

$$= x \sin x - \underbrace{a \sin a} - \sin x - \underbrace{\sin a}$$

$$= \left(x \sin x - \sin x \right) + k$$