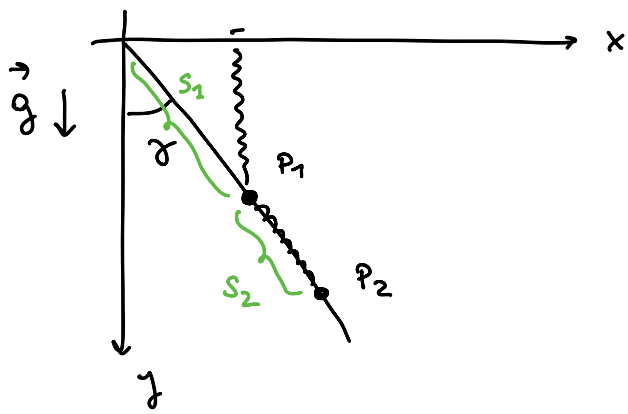
Lesson 28 - 01/12/2022



γ constant

k

m

L ?

$\vec{OP}_1 = (s_1 \sin\gamma, \; s_1 \cos\gamma) \Rightarrow \vec{v}_1 = (\dot{s}_1 \sin\gamma, \; \dot{s}_1 \cos\gamma)$

$\vec{OP}_2 = ((s_1+s_2)\sin\gamma, \; (s_1+s_2)\cos\gamma) \Rightarrow \vec{v}_2 = ((\dot{s}_1+\dot{s}_2)\sin\gamma,$
$(\dot{s}_1+\dot{s}_2)\cos\gamma).$

$\Downarrow$

$$K = \frac{1}{2} m \left(2\dot{s}_1^2 + \dot{s}_2^2 + 2\dot{s}_1\dot{s}_2 \right)$$

$$V = -mgy_1 - mgy_2 + \frac{1}{2}k y_1^2 + \frac{1}{2}k|P_1 - P_2|^2$$
$$= -2mg\, s_1\cos\gamma - mg\, s_2 \cos\gamma + \frac{1}{2}k(s_1\cos\gamma)^2 + \frac{1}{2}k s_2^2$$

Equilibria ?

$$\begin{cases} \partial V/\partial s_1 = -2mg\cos\gamma + k s_1 \cos^2\gamma = 0 \\ \partial V/\partial s_2 = -mg\cos\gamma + k s_2 = 0 \end{cases}$$

$$\bar{s}_1 = \frac{2mg}{k\cos\gamma}, \quad \bar{s}_2 = \frac{mg\cos\gamma}{k}$$

Stability ?

$$\text{Hess } V(s_1,s_2) = \begin{pmatrix} k\cos^2\gamma & 0 \\ 0 & k \end{pmatrix} \in \text{Sym}^+ \Rightarrow (\bar{s}_1, \bar{s}_2) \text{ is stable.}$$

$$Q(s_1,s_2) = m\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Frq. of small oscillations around $(\bar{s}_1, \bar{s}_2)$

$$\det\left(\text{Hess}\, V(\bar{s}_1,\bar{s}_2) - \omega^2 Q(\bar{s}_1,\bar{s}_2)\right) = 0$$

That is :

$$\det\begin{pmatrix} k\cos^2\gamma - 2\omega^2 m & -\omega^2 m \\ -\omega^2 m & k - \omega^2 m \end{pmatrix} = 0$$

$$(\ldots) \qquad \omega_{1,2}^2 = \frac{2mk + mk\cos^2\gamma \pm \sqrt{4m^2k^2 + m^2k^2\cos^4\gamma}}{2m^2}$$

$\omega_{1,2} \to$ positive root of $\omega_{1,2}^2$ respectively.

—x—x—

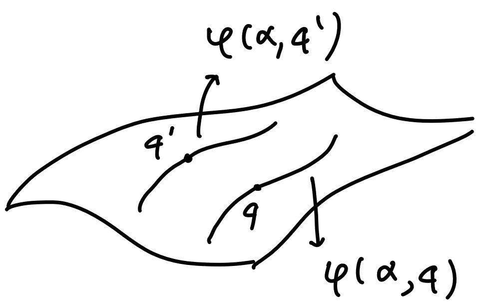
Noether theorem

$L(q,\dot{q},t)$, we consider a transformation

$$q \longmapsto \varphi(\alpha, q) \qquad \alpha \in (-A, A)$$

such that

$$\varphi(0,q) = \text{id}(q) = q.$$



$\forall q$, $\varphi(\alpha,q)$ is a piece of curve through q

we can complete the function $q \mapsto \varphi(\alpha,q)$ to the phase-space by

$$\dot q \longmapsto \psi(\alpha, q, \dot q) = \frac{d\varphi}{dt}(\alpha, q)$$

That is

$$\psi_h(\alpha, q, \dot q) = \sum_{k=1}^{n} \frac{\partial \varphi_h}{\partial q_k}(\alpha, q)\, \dot q_k$$

Remark

$\psi(0, q, \dot q) = id(\dot q) = \dot q$

since the Jacobian matrix $\dfrac{\partial \varphi_h}{\partial q_k}$ for $\alpha = 0$ is the identity matrix (by assumption).

Therefore:

$(q, \dot q) \longmapsto (\varphi(\alpha, q), \psi(\alpha, q, \dot q))$ is a piece of curve through $(q, \dot q)$.

Special case (of Noether theorem, below ..)

$L(q, \dot q, t)$ admits a cyclic coordinate. $\underline{\underline{q_e}}$

$\varphi_h(\alpha, q) = q_h + \alpha\, \delta_{he} \longrightarrow \alpha$ translation of q_e coo.

$\psi_h(\alpha, q, \dot q) = \dot q_h$

$L(q, \dot q, t) = L(\varphi(\alpha, q), \psi(\alpha, q, \dot q), t)$

we know that, in such simplest case, there is a conserved quantity:

$$\boxed{p_e = \frac{\partial L}{\partial \dot q_e}}$$

Noether theorem

Let

$$(q,\dot q) \longmapsto (\varphi(\alpha,q), \psi(\alpha,q,\dot q))$$

$\alpha \in (-A, A)$. (as above).

IF

$$\boxed{L(\varphi(\alpha,q), \psi(\alpha,q,\dot q), t) = L(q,\dot q,t)}$$

$\forall\ (q,\dot q, t)$ and $\forall\ \alpha \in (-A, A)$.

THEN

$$P(q,\dot q,t) = \sum_{h=1}^{n} \frac{\partial \varphi_h}{\partial \alpha}(0,q) \underbrace{P_h(q,\dot q,t)}_{= \frac{\partial L}{\partial \dot q_h}(q,\dot q,t)}$$

is a constant of motion for $L(q,\dot q,t)$.

Remark

For the case of a cyclic coo. we clearly obtain $P = p_e$.

In general, P is a linear combination of momenta $\frac{\partial L}{\partial \dot q_h}$ with coeff. $\frac{\partial \varphi_h}{\partial \alpha}(0,q)$.

Proof

$$L(q,\dot q,t) = L(\varphi(\alpha,q), \psi(\alpha,q,\dot q), t)$$

$$\Downarrow$$

The derivative w.r.t. α of the 2nd member is $\equiv 0$.

$$\sum_{h=1}^{m}\Big[\frac{\partial L}{\partial q_h}\big(\varphi(\alpha,q),\psi(\alpha,q,\dot q),t\big)\frac{\partial \varphi_h}{\partial\alpha}(\alpha,q) + \frac{\partial L}{\partial \dot q_h}\big(\varphi(\alpha,q),\psi(\alpha,q,\dot q),t\big)\frac{\partial\psi_h}{\partial\alpha}(\alpha,q,\dot q)\Big]=0.$$

From the def: $\psi_h = \frac{d\varphi_h}{dt} \Rightarrow$

$$\frac{\partial\psi_h}{\partial\alpha} = \frac{\partial}{\partial\alpha}\frac{d\varphi_h}{dt} = \frac{d}{dt}\frac{\partial\varphi_h}{\partial\alpha}$$ (proven during Lagr. eq.)

We sub. this expression and we evaluate in $\alpha=0$, obtaining:

$$\sum_{h=1}^{m}\Big[\frac{\partial L}{\partial q_h}(q,\dot q,t)\frac{\partial \varphi_h}{\partial\alpha}(0,q) + \frac{\partial L}{\partial \dot q_h}(q,\dot q,t)\frac{d}{dt}\frac{\partial\varphi_h}{\partial\alpha}(0,q)\Big]=0$$

But - by Lagrange eqs - along solutions we have that $\frac{\partial L}{\partial q_h} = \frac{d}{dt}\frac{\partial L}{\partial \dot q_h}$

$$\sum_{h=1}^{m}\Big[\frac{d}{dt}\frac{\partial L}{\partial \dot q_h}\frac{\partial \varphi_h}{\partial\alpha}(0,q) + \frac{\partial L}{\partial \dot q_h}(q,\dot q,t)\frac{d}{dt}\frac{\partial\varphi_h}{\partial\alpha}(0,q)\Big]=0$$

That is

$$\frac{d}{dt}\Big[\sum_{h=1}^{m}\frac{\partial L}{\partial \dot q_h}(q,\dot q,t)\frac{\partial \varphi_h}{\partial\alpha}(0,q)\Big]=0$$

$\Leftarrow$)

$$P(q,\dot q,t) := \sum_{h=1}^{n} \frac{\partial \varphi_h}{\partial \alpha}(0,q)\frac{\partial L}{\partial \dot q_h}(q,\dot q,t)$$

is a first integral for L.

$$P(q,\dot q,t) := \sum_{h=1}^{n} \frac{\partial \varphi_h}{\partial \alpha}(0,q)\, p_h(q,\dot q,t)$$

$\square$

Emmy Noether $\longrightarrow$ See video on line

A fundamental example

Central potential

m_1, m_2, no constraints

Central potential: V depends only on the distance $|P_2 - P_1|$ of points.

$\vec{OP_1} = (q_1, q_2, q_3)$

$\vec{OP_2} = (q_4, q_5, q_6)$

$$L(q,\dot q) = \frac{1}{2}m_1(\dot q_1^2 + \dot q_2^2 + \dot q_3^2) + \frac{1}{2}m_2(\dot q_4^2 + \dot q_5^2 + \dot q_6^2) - V\left((q_4-q_1)^2 + (q_5-q_2)^2 + (q_6-q_3)^2\right)$$

we prove - by Noether theo - that:

TOTAL QUANTITY of MOTION a)

&

TOTAL ANGULAR MOMENTUM b)

are conserved quantities.

a) L is invariant with respect to translations along any cartesian axis. For ex. consider the translation w.r.t. the x-axis:

$$q_h \longmapsto \varphi_h(\alpha, q),$$
$$\dot{q}_h \longmapsto \psi_h(\alpha, q, \dot{q}).$$

$\varphi_1 = q_1 + \alpha$, $\varphi_4 = q_4 + \alpha$

$\varphi_h = q_h \qquad \forall\, h \neq 1, 4.$ Corresp. cons. quantity?

$$P = \underset{\frac{\partial \mathcal{L}}{\partial \dot{q}_1}}{\underset{\shortparallel}{p_1}} + \underset{\frac{\partial \mathcal{L}}{\partial \dot{q}_4}}{\underset{\shortparallel}{p_4}} = m_1 \dot{q}_1 + m_2 \dot{q}_4 =$$ first component of the vector "quantity of motion".

→ Same argument for translations w.r.t y axis and z axis

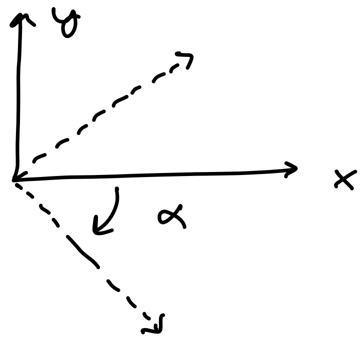
y axis ↓ 2nd component of quantity of motion.

z axis ↓ last component of quantity of motion.

⟹ QUANTITY of MOTION IS CONSERVED.

b) L is invariant under rotations around any cartesian axis. For ex. consider rotations

Round 2



$q_n \longmapsto \varphi_n(q, \alpha)$ def. as :

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\varphi_3 = q_3$$

$$\begin{pmatrix} \varphi_4 \\ q_5 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_4 \\ q_5 \end{pmatrix}$$

$$\varphi_6 = q_6$$

with natural extensions to $\dot{q}_n \mapsto \varphi_n(\alpha, q, \dot{q})$

Clearly, L is invariant w.r.t. the rotation above. And :

$$\frac{\partial \varphi_1}{\partial \alpha}(0, q) = (-\sin\alpha\, q_1 - \cos\alpha\, q_2)\big|_{\alpha=0} = -q_2$$

$\downarrow$

$\varphi_1 = \cos\alpha\, q_1 - \sin\alpha\, q_2$

$$\frac{\partial \varphi_2}{\partial \alpha}(0, q) = (\cos\alpha\, q_1 - \sin\alpha\, q_2)\big|_{\alpha=0} = q_1$$

$\downarrow$

$\varphi_2 = \sin\alpha\, q_1 + \cos\alpha\, q_2$

$$\frac{\partial \varphi_3}{\partial \alpha}(0, q) = 0$$

$$\overset{(!)}{\triangleleft} \quad -q_2 p_1 + q_1 p_2 = -q_2 (m_1 \dot{q}_1) + q_1 (m_2 \dot{q}_2)$$

Analogously, for $\frac{\partial \varphi_4}{\partial \alpha}$, $\frac{\partial \varphi_5}{\partial \alpha}$, $\frac{\partial \varphi_6}{\partial \alpha}$

$$-q_5 p_4 + q_4 p_5$$

$$\Rightarrow P = \underbrace{m_1 (q_1 \dot{q}_2 - q_2 \dot{q}_1) + m_2 (q_4 \dot{q}_5 - q_5 \dot{q}_4)}$$

$\parallel$

third component of the vector "total angular momentum"

$\downarrow$

$$m_1 (q_1, q_2, q_3) \wedge (\dot{q}_1, \dot{q}_2, \dot{q}_3) + m_2 (q_4, q_5, q_6) \wedge (\dot{q}_4, \dot{q}_5, \dot{q}_6)$$

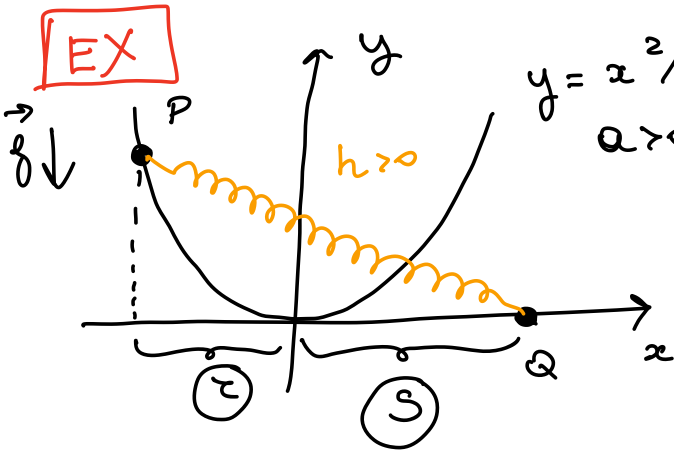
$\Downarrow$

TOTAL ANGULAR MOMENTUM IS CONSERVED

Moral By the Lagrangian formalism, we have proven that PHYSICS CONSERVATION LAWS CAN BE OBTAINED FROM "SYMMETRIES" (FROM the invariance of L from families of transformations, e.g. translations, rotations...)

The conservation of energy (remind Jacobi integral...!) comes from the invariance of L under temporal translations. Indeed $\dot{E} = \dots = -\frac{\partial L}{\partial t}$

$\boxed{EX}$



$y = x^2/a$
$a>0$

P, m
Q, m

$\tau = x_P$
$h = mg/a$

- $L(\tau, s, \dot{\tau}, \dot{s})$
- $h = mg/a$,
- $a = a(\tau, s)$?

Freq. of small oscillations around the stable eq.

$$V_{el} = \frac{1}{2} k \left((-\tau + s)^2 + \frac{\tau^4}{a^2} \right)$$

$\vec{OP} = (\tau, \tau^2/a)$

$\vec{OQ} = (s, 0)$

—x—x—