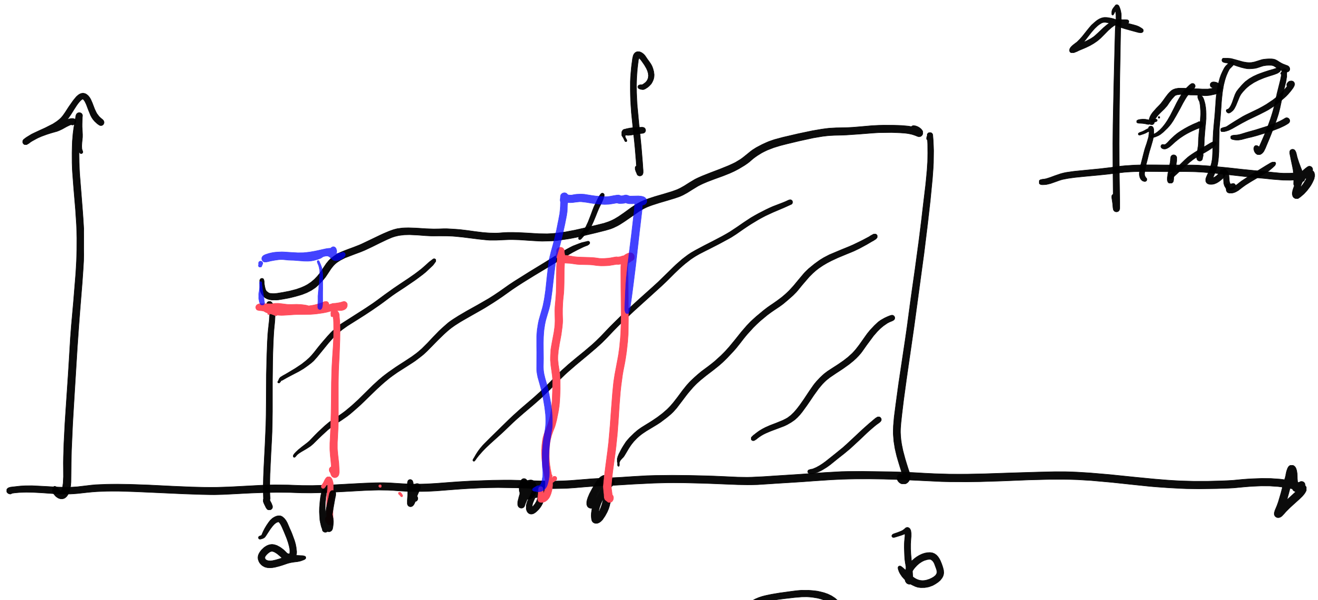


$$f : [a,b] \longrightarrow \mathbb{R}$$

$$f(x) \geq 0 \quad \forall x \in [a,b]$$

$$\underline{A}(f) = \sup_{\pi} \underline{S}_f(\pi)$$

π subdivision of $[a,b]$

$$\overline{A}(f) = \inf_{\pi \ldots} \overline{S}_f(\pi)$$

$$\left(\underline{S}_f(\pi) = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, m_k \right.$$

$$\pi = \{a = x_0, x_1, \ldots, x_n = b\}$$

$$m_k = \inf_{[x_k, x_{k+1}]} f$$

$$\overline{S}_f = \sum_{k=0}^{n-1} (x_{k+1} - x_k) M_k$$

$$M_k = \sup_{[x_k, x_{k+1}]} f$$

$A(f)$ = Area of f

if

$$\underline{A}(f) = \overline{A}(f) = A(f)$$

Proposition: Let $f : [a,b] \to \mathbb{R}$ bounded function

(i.e. $\exists\, K$ s.t. $|f(x)| \leq K \quad \forall x \in [a,b]$

$A(f)$ is well-defined

$\underbrace{\text{if and only if}}_{\text{iff}}$

$\underline{\forall \varepsilon > 0}$ $\exists$ π^+, π^- subdivision

$$0 \leq \underline{\overline{S}(\pi^+)} - \underline{\underline{S}(\pi^-)} < \varepsilon$$

<u>Corollary</u>. Same hypotheses as above, assume that $\forall_n$

π_n and

$$\lim_{n \to +\infty} \underbrace{\underline{S}_f(\pi_n)} = \lim_{n \to \infty} \overline{S}_f(\pi_n)$$

then $A(f)$ is well-def

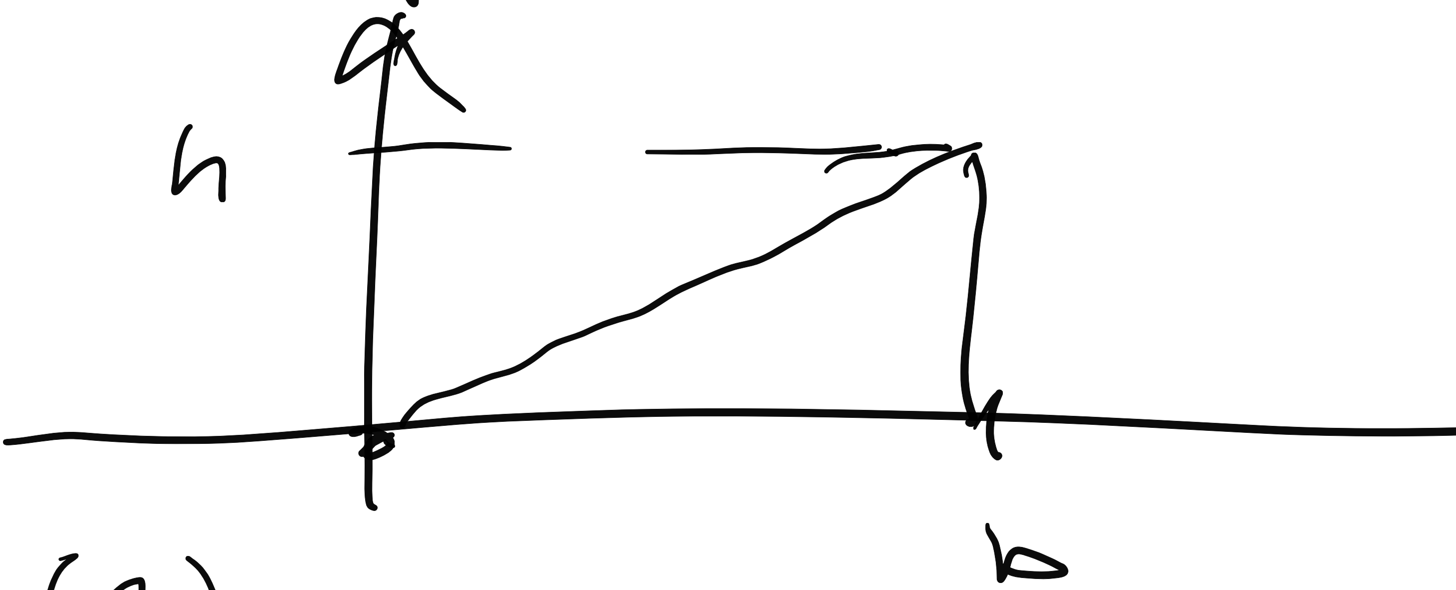
i.e $\underline{A}(f) = \overline{A}(f)$

and $A(f) = \lim_{n \to +\infty} \underline{S}_f(\pi_n) =$

$= \lim \overline{S}_f(\pi_n)$

The converse is also true.

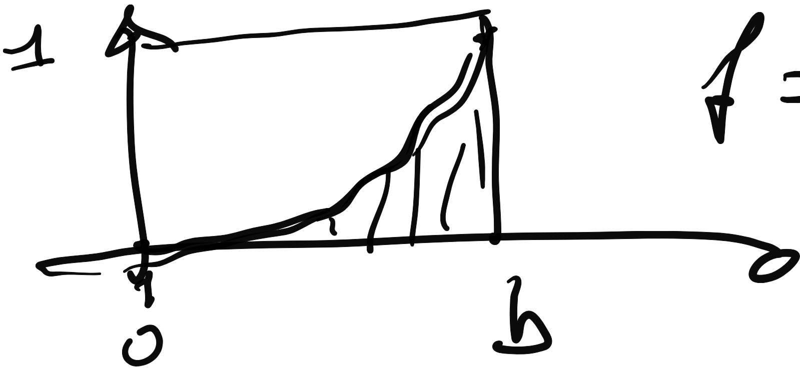
We used it yesterday to compute the area of



$\underline{S}(\pi_n)$

$\longrightarrow \frac{hb}{2}$

$\overline{S}(\pi_n)$



$f = x^2$

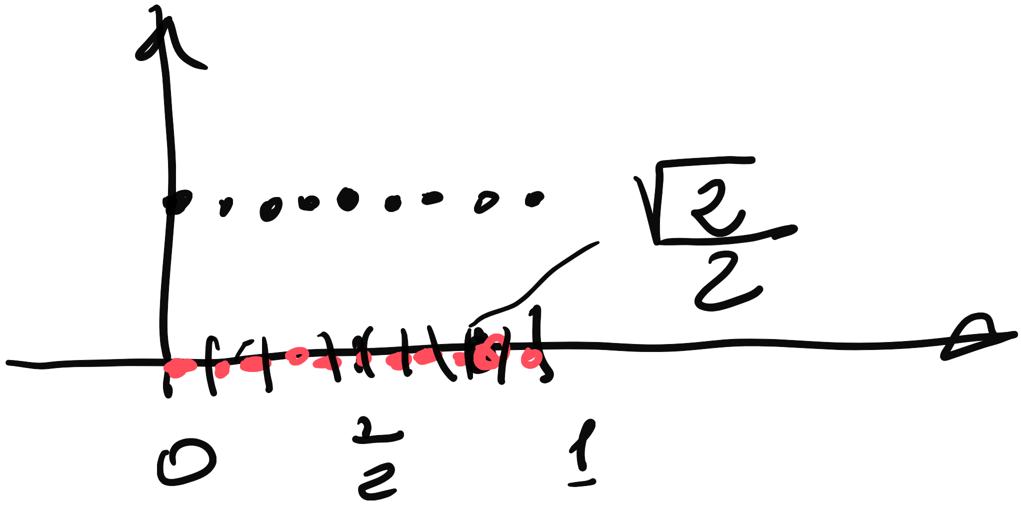
$$\frac{b^3}{3}$$

Is it possible that

$$\underline{A}(f) < \overline{A}(f)$$

$$f : [0 \;\; 1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



$$\underline{S}_f = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, m_k = 0$$

$$m_k = \inf_{[x_k, x_{k+1}]} = 0$$

$$\overline{S}_f = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, M_k = \sum_{k=0}^{n-1} (x_{k+1} - x_k) = 1$$

$M_k = 1$

$\forall$ any $(\Pi_n)_{n \in \mathbb{N}}$ subdivisions

$$\overline{S}_f(\Pi_n) - \underline{S}_f(\Pi_n) = 1$$

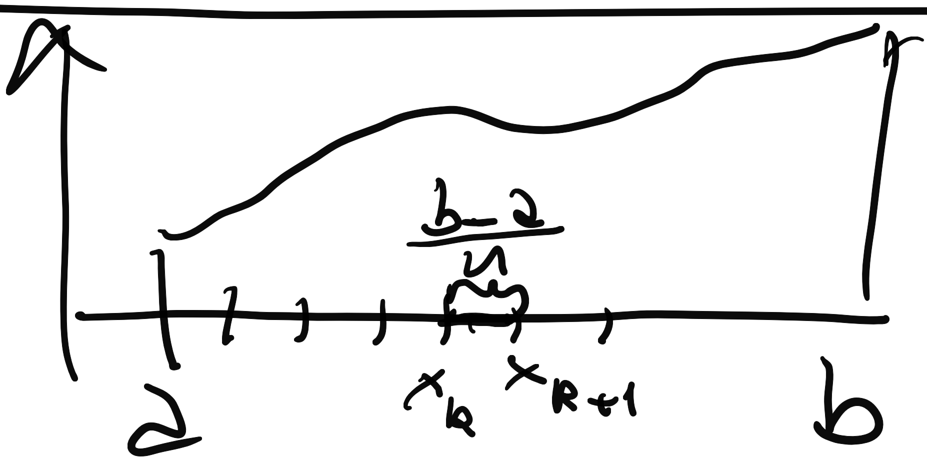
Theorem if $f \in \mathcal{C}([a,b])$ (i.e. $f:[a,b] \to \mathbb{R}$ is CONTINUOUS)

$f(x) \geq 0$, then

$A(f)$ is well defined

$$\left(\Longleftrightarrow \underline{A}(f) = \overline{A}(f) = A(f) \right)$$

Proof



$$\left\{ x_0 = a \qquad x_k = a + k\frac{b-a}{n} \right\} = \pi_n$$

$$\underline{S}_f(\pi_n) = \sum (x_{k+1} - x_k) \cdot m_k$$

$$m_k = \inf_{[x_k, x_{k+1}]} f(x) = \min_{[x_k, x_{k+1}]} f(x)$$

$$\parallel$$

$$f(\xi_k)$$

by Weierstrass theorem.

$$\overline{S}_f(\pi_n) = \sum \underbrace{(x_{k+1} - x_k)}_{\frac{b-a}{n}} M_k$$

$$M_k = \max_{[x_k, x_{k+1}]} f(x) = f(\eta_k)$$

$$\overline{S}_f(\pi_n) - \underline{S}_f(\pi_n) =$$

$$\sum_{k=0}^{n-1} \frac{b-a}{n} (M_k - m_k) =$$

$$\boxed{\frac{b-a}{n} \sum_{k=0}^{n-1} f(\eta_k) - f(\xi_k)} =$$

with $\eta_k, \xi_k \in [x_k, x_{k+1}]$

Add the hypothesis that $f \in \mathcal{C}^1([a,b])$, i.e.

$\exists f'(x)\ \forall x \in [a,b]$ and

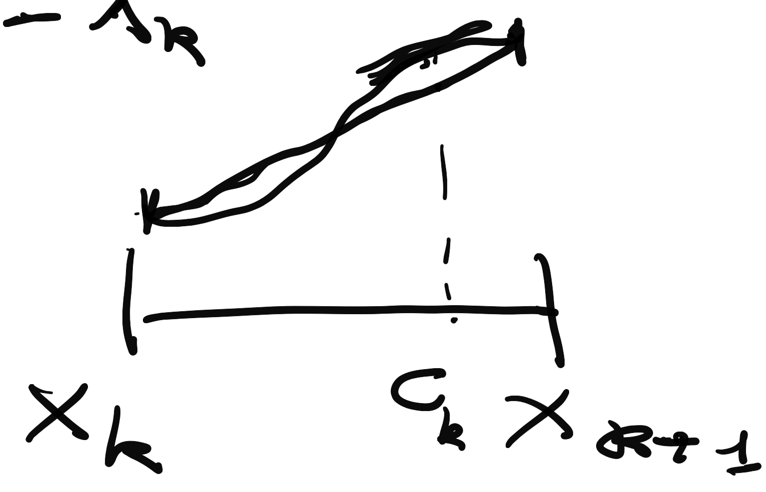
$f' : [a,b] \to \mathbb{R}$ is

continuous.

In the interval $[x_k, x_{k+1}]$

we apply

$$\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = f'(c_k)$$



$$= \frac{b-a}{n}\sum_{k=0}^{n-1} f'(c_k)\underbrace{(x_{k+1}-x_k)}_{\frac{b-a}{n}}$$

$$= \frac{(b-a)^2}{n^2}\sum_{k=0}^{n-1} f'(c_k) \le *$$

But f' is continuous

~~the~~ so $|f'|$ is continuous

$\exists\, M$ s.t. $|f'(x)| \le M \quad \forall x \in [a,b]$

$$\le \left|\frac{(b-a)^2}{n^2}\sum_{k=0}^{n-1} f'(c_k)\right| \le$$

$$\le \sum \left|\left(\frac{b-a}{n^2}\right) f'(c_k)\right| \le \frac{b-a}{n^2}\sum |f'(c_k)|$$

$$\le \frac{b-a}{n^2}\sum_{k=0}^{n-1} M = \frac{b-a}{n^2} M \sum_{k=0}^{n-1} = \frac{b-a}{n^2} M n$$

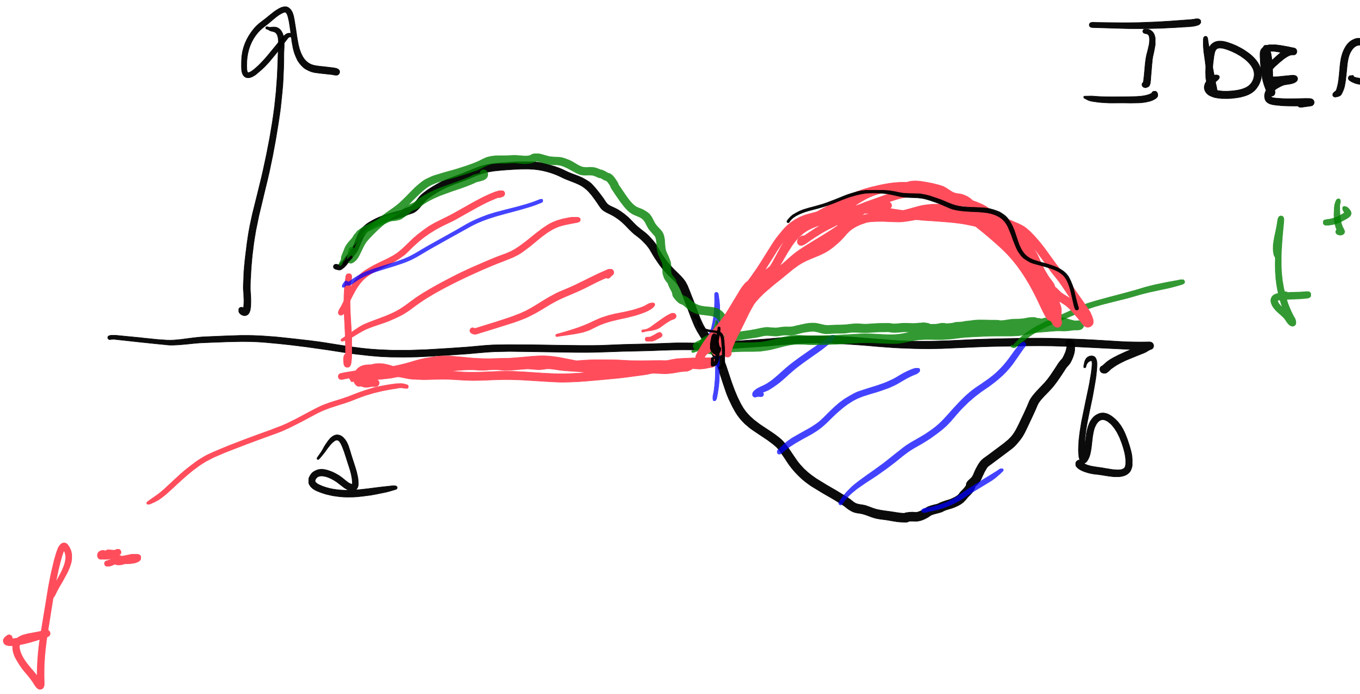
$$\Longrightarrow \lim \underline{S}(\pi_n) =$$

$$= \lim \overline{S}(\pi_n) =$$

$$= A(f)$$

Thanks to corollary.

IDEA



$$f : [a, b] \longrightarrow \mathbb{R}$$

<u>Define</u>

$$f^{+} : [a, b] \longrightarrow \mathbb{R}$$

$$f^{+}(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

" POSITIVE PART" of f

Also

$$f^{-}(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

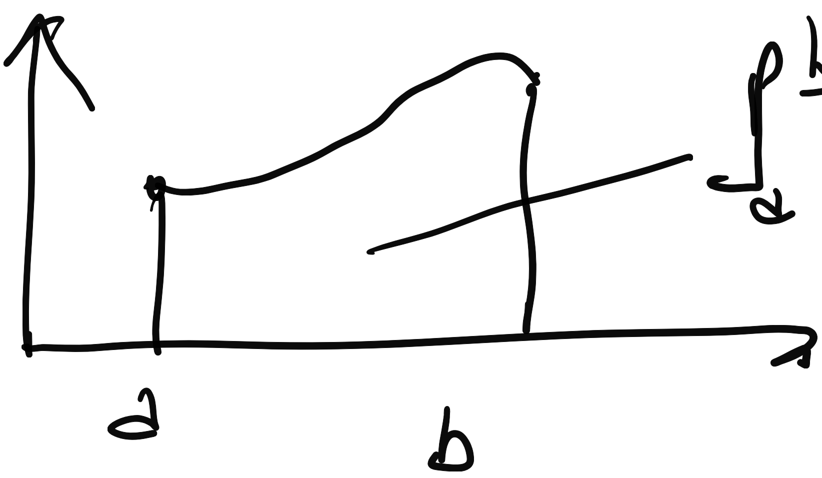
Observe that

$$f^{+}(x) \geq 0 \quad f^{-}(x) \geq 0$$

$$\forall x \in [a, b]$$

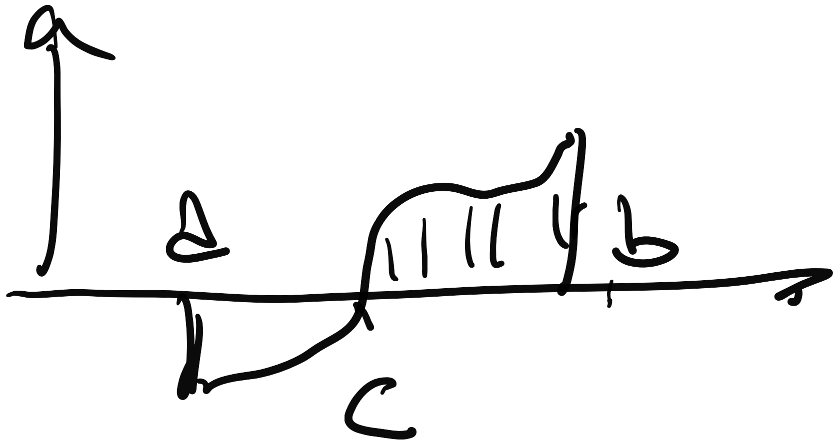
Definition: $f:[a,b]\to\mathbb{R}$, bounded, we say that it is integrable, if f^+ and f^- have well defined area, and we set

$$\int_a^b f(x)\,dx := A(f^+) - A(f^-)$$



$$\int_a^b f\,dx = A(f)$$

$$\int_a^b f\,dx = -A(f)$$



$$\int_a^b f\,dx = \underset{[c,b]}{\mathrm{Area}}(f) - \underset{[a,c]}{\mathrm{Area}}(-f)$$

<u>Corollary</u>: $f \in \mathcal{C}([a,b])$

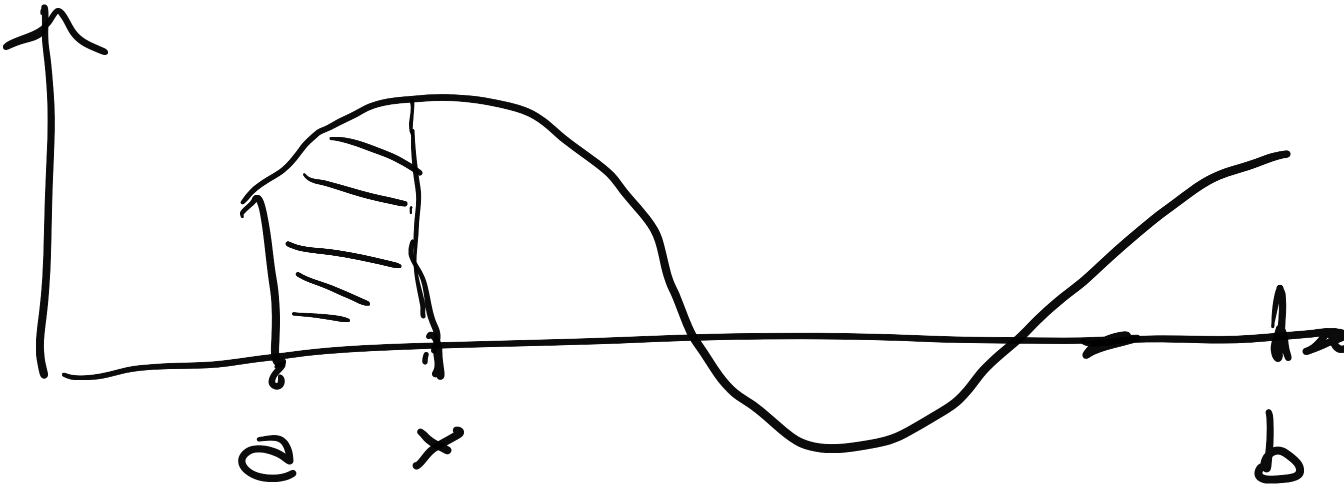
f is integrable.

- It is also true that every monotone f is integrable

Suppose $f : [a,b] \to \mathbb{R}$ is integrable.

Define $F : [a,b] \to \mathbb{R}$

$$F(x) = \int_a^x f(t)\,dt$$



$$F(x) = \int_a^x f(t)\,dt$$

Theorem (Found. Th of Integral Calculus)

$$F'(x) = f(x)$$