Lesson 27 - 30/11/2022

- Potentials dep. on velocities
- EX 1
- EX 2

Let consider:

$\vec{F} = -2m\vec{\omega} \wedge \dot{\vec{q}}$ (Coriolis force) → a point of mass m in a rotating system with angular velocity $\vec{\omega}$.

$\vec{F} = e(\vec{E} - \vec{B} \wedge \dot{\vec{q}})$ (Lorentz force) ↓ determine the dynamics of an electric charge in a given electromagnetic field.

depending on $\dot{\vec{q}}$.

↓ Lagrange eqs. in the general form:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_K} - \frac{\partial K}{\partial q_K} = Q_h \quad (1) \qquad \text{where} \quad Q_h = \sum_{i=1}^{m} \vec{F}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h}$$

(not Lagrange eqs in the form $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_h} - \frac{\partial \mathcal{L}}{\partial q_h} = 0$ (2))

In this lecture we prove that for these forces the form (2) is maintained, since $\exists\, V_1(q, \dot{q})$ ("generalized potential") such that: $\mathcal{L} = K - V_1$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_h}\right) - \frac{d}{dt}\left(\frac{\partial V_1}{\partial \dot{q}_h}\right) - \frac{\partial K}{\partial q_h} + \frac{\partial V_1}{\partial q_h} = 0 \\ \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_h}\right) - \frac{\partial K}{\partial q_h} = Q_h \end{cases}$$

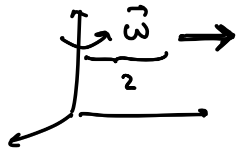
$$\Leftrightarrow \boxed{Q_h = \frac{d}{dt}\left(\frac{\partial V_1}{\partial \dot{q}_h}\right) - \frac{\partial V_1}{\partial q_h}} \longrightarrow (*)$$

— x — x —

We check that both the Coriolis force and the Lorentz force admit a "generalized" potential.

<u>CORIOLIS FORCE</u> <u>with $\vec{\omega}$ a constant vector</u> In this non-inertial (rotating) system:

- $\vec{F}_{cf} = m\omega^2 r\, \vec{e}_r$



$\Downarrow$ is a conservative force

$V_{cf} = -\frac{1}{2} m\omega^2 r^2$

- $\vec{F} = -2m\vec{\omega}\wedge\dot{\vec{q}}$

Prop $L(\vec{q},\dot{\vec{q}}) = \frac{1}{2} m \dot{\vec{q}}^{\,2} - V_{cf} - V_1$

where $V_1 = m\,\vec{\omega}\wedge\vec{q}\cdot\dot{\vec{q}} = m\,\vec{q}\wedge\vec{\omega}\cdot\dot{\vec{q}}$

Proof We need to prove that formula $(*)$ holds.

$$\frac{\partial V_1}{\partial \dot{q}_h} = m(\vec{q}\wedge\vec{\omega})_h \Rightarrow \frac{d}{dt}\left[m(\vec{q}\wedge\vec{\omega})\right]_h = $$

$$= \underline{m(\dot{\vec{q}}\wedge\vec{\omega})_h}$$

$$\frac{\partial V_1}{\partial q_h} = m(\vec{\omega}\wedge\dot{\vec{q}})_h$$

$$\Rightarrow \frac{d}{dt}\frac{\partial V_1}{\partial \dot{q}_h} - \frac{\partial V_1}{\partial q_h} = m(\dot{\vec{q}}\wedge\vec{\omega})_h - m(\vec{\omega}\wedge\dot{\vec{q}})_h =$$

$$= -2m(\vec{\omega}\wedge\dot{\vec{q}})_h = Q_h \quad \boxtimes$$

LORENTZ FORCE

$$\vec{F} = e(\vec{E} - \vec{B}\wedge\dot{\vec{q}})$$

If $\vec{B}$ is constant and uniform $\Rightarrow -e\vec{B}\wedge\dot{\vec{q}}$ is a term like the one of the Coriolis's force.

$\Downarrow$

The "generalized" potential is

$$\boxed{\frac{1}{2} e\vec{B}\wedge\dot{\vec{q}}\cdot\vec{q}}$$

Moreover $\vec{E}$ comes from a potential ϕ such that

$\vec{E} = -\nabla\phi$:

$$V_1(\vec{q},\dot{\vec{q}}) = \frac{1}{2} e \vec{B} \wedge \dot{\vec{q}} \cdot \vec{q} + e\phi$$

— x — x —

It is possible to treat also the general case $\vec{E}(\vec{q},t)$ and $\vec{B}(\vec{q},t)$.

But we need to use the "vector potential" $\vec{A}$ such: (a conseq. of Maxwell eqs...)

$$\begin{cases} \vec{E} = -\left(\nabla\phi + \dfrac{\partial \vec{A}}{\partial t}\right) \\ \vec{B} = \nabla \wedge \vec{A} \end{cases}$$

Prop In the general case, the Lorentz force admits a "generalized" potential :

$$V_1(\vec{q},\dot{\vec{q}}) = e\phi - e\dot{\vec{q}} \cdot \vec{A}.$$

Proof

$$\vec{F} = \underbrace{-e\underbrace{\left(\nabla\phi + \frac{\partial \vec{A}}{\partial t}\right)}_{-\vec{E}}}_{e\vec{E}} - e \underbrace{\nabla\wedge\vec{A}}_{\vec{B}} \wedge \dot{\vec{q}} =$$

$$= -e\left(\nabla\phi + \frac{\partial \vec{A}}{\partial t}\right) + e\dot{\vec{q}} \wedge \nabla \wedge \vec{A}$$

Now, let calculate $\boxed{\dfrac{d}{dt}\dfrac{\partial V_1}{\partial \dot{q}_n} - \dfrac{\partial V_1}{\partial q_n}}$

$$\frac{\partial V_1}{\partial \dot{q}_n} = -eA_n \Rightarrow \frac{d}{dt}\frac{\partial V_1}{\partial \dot{q}_n} = -e\left(\frac{\partial A_n}{\partial t} + \sum_{k=1}^{3}\frac{\partial A_n}{\partial q_k}\dot{q}_k\right)$$

$$\frac{\partial V_1}{\partial q_h} = \underline{e\left(\frac{\partial \phi}{\partial q_h} - \sum_{k=1}^{3} \dot q_k \frac{\partial A_k}{\partial q_h}\right)}$$

$\Rightarrow$

$$\frac{d}{dt}\frac{\partial V_1}{\partial \dot q_h} - \frac{\partial V_1}{\partial q_h} =$$

$$= -e\left(\overline{\frac{\partial A_h}{\partial t}} + \sum_{k=1}^{3}\frac{\partial A_h}{\partial q_k}\dot q_k\right) - e\left(\overline{\frac{\partial \phi}{\partial q_h}} - \sum_{k=1}^{3}\dot q_k \frac{\partial A_k}{\partial q_h}\right)$$

$$= -e\left(\frac{\partial \phi}{\partial q_h} + \frac{\partial A_h}{\partial t}\right) + e\sum_{k=1}^{3}\dot q_k\left(\frac{\partial A_k}{\partial q_h} - \frac{\partial A_h}{\partial q_k}\right) =$$

calculation (easy)

$$= -e\left(\frac{\partial \phi}{\partial q_h} + \frac{\partial A_h}{\partial t}\right) + e\left(\dot{\vec q}\wedge \nabla \wedge \vec A\right)_h = Q_h$$

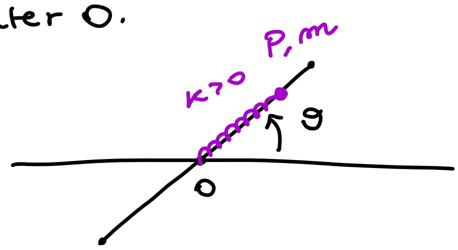
$\square$

—x—x—x—

EX 1

Bar AB, m, ℓ.

Baricenter O.



r = distance between O and P.

r, θ = Lagr. coordinates.

- L + eqs. of motion
- L_R, its equilibria, stability. (Eventually) freq. of small oscillations. Bif. diagram.
- Original system: eq., stability.

Lagrangian

$\vec{OP} = (r\cos\theta, r\sin\theta) \Rightarrow \vec v_P = (\dot r\cos\theta - r\dot\theta\sin\theta,$
$\dot r\sin\theta + r\dot\theta\cos\theta)$

$$\Rightarrow |\vec{v}_P|^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}\left[m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m e^2}{12}\dot{\theta}^2\right] - \frac{1}{2}k r^2$$

$$= \frac{1}{2}\left[m\dot{r}^2 + \left(m r^2 + \frac{m e^2}{12}\right)\dot{\theta}^2\right] - \frac{1}{2}k r^2$$

Eqs of motion :

$$\begin{cases} m\ddot{r} = m r\dot{\theta}^2 - k r \\ \left(m r^2 + \frac{m e^2}{12}\right)\ddot{\theta} + 2 m r\dot{r}\dot{\theta} = 0 \end{cases}$$

$\downarrow$

θ is a cyclic coo.

L_R

$$L = L(r,\dot{\theta},\dot{r}) \underset{1\text{-dim}}{\Rightarrow} \frac{\partial L}{\partial\dot{\theta}} = \left(m r^2 + \frac{m e^2}{12}\right)\dot{\theta} = J$$

$$\Rightarrow \dot{\theta} = J\Big/\left(m r^2 + \frac{m e^2}{12}\right)$$

$$L_R(r,\dot{r}) = \frac{1}{2}\left[m\dot{r}^2 + \underbrace{\cancel{\left(m r^2 + \frac{m e^2}{12}\right)}\frac{J^2}{\left(m r^2 + \frac{m e^2}{12}\right)^{\cancel{2}}}}\right] \underbrace{- \frac{1}{2}k r^2 - \frac{J\,J}{m r^2 + \frac{m e^2}{12}}}$$

$$= \frac{1}{2}m\dot{r}^2 - V_R(r)$$

where

$$V_R(r) = \frac{1}{2}k r^2 + \frac{1}{2}\frac{J^2}{(m r^2 + m e^2/12)}$$

$\rightarrow +\infty$ when $r \rightarrow \pm\infty$. Equilibria ?

$$V_R'(z) = Kz - \frac{mzJ^2}{(mr^2 + me^2/12)^2} = 0$$

$$\Leftrightarrow z\underbrace{\left(K - \frac{mJ^2}{(mr^2+me^2/12)^2}\right)} = 0$$

(a) $z = 0$

(b) $K\left(mz^2 + \frac{me^2}{12}\right)^2 - mJ^2 = 0$

$$\Leftrightarrow K\left(mr^2 + \frac{me^2}{12}\right)^2 = mJ^2$$

(....)

$$z^2 = \underbrace{-\frac{e^2}{12} + J\sqrt{\frac{1}{km}}}_{\text{only when } > 0}$$

that is when $J \geq \frac{e^2\sqrt{km}}{12}$: other 2 equilibria $\pm z^*$ $(z^* > 0)$.

$$\boxed{I := \frac{e^2 m}{12}}$$

the previous cond. becomes

$$\boxed{J \geq I\sqrt{\frac{k}{m}}}$$

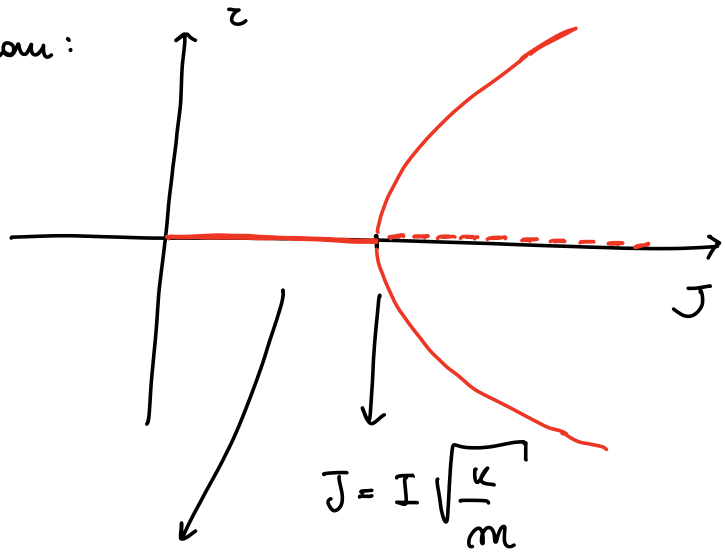
Stability ?!

$V_R''(z) = \dots$

Conclusion $\nearrow$ 0 is stable when $J < I\sqrt{\frac{k}{m}}$
unstable otherwise.

Equilibria $\pm z^*$ are (when exist) <u>STABLE</u>.

Bif. diagram:

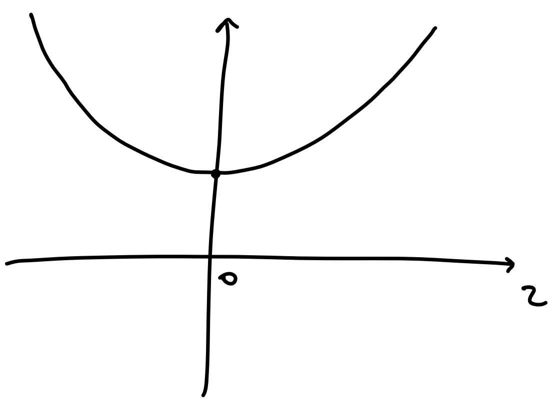


$J = I\sqrt{\frac{k}{m}}$

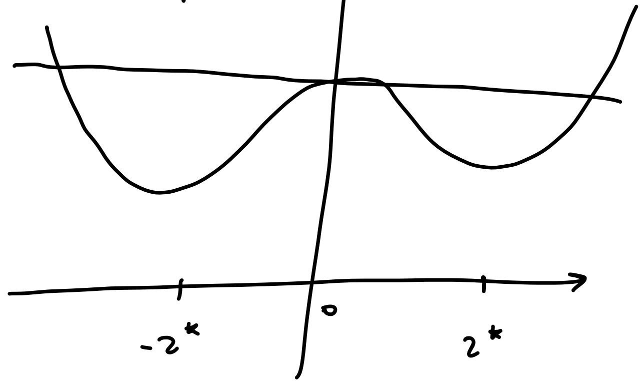
freq. of small oscillation : $\omega^2 = \dfrac{V_R''(0)}{m}$

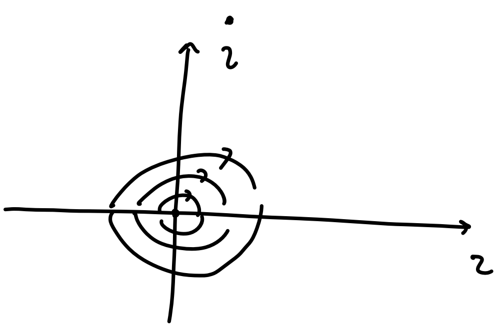
Phase-portrait of the reduced system.

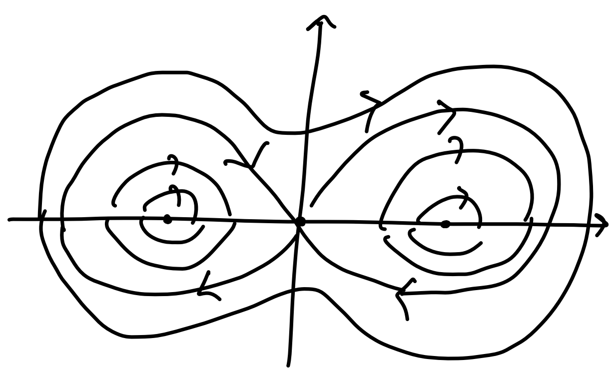
$J < \sqrt{\frac{k}{m}}\, I$



$J > \sqrt{\frac{k}{m}}\, I$







Eq. of original system.

$V(z) = \frac{1}{2} k z^2.$

$z = 0$

$(0, \vartheta^*, 0, 0)$, $\forall \vartheta^* \in [0, 2\pi)$.

$\operatorname{Hess} V(z, \vartheta) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ semi-positive def.

Stability must be checked directly.

$(0, \vartheta^*, 0, \varepsilon)$, ε small.

$(z(t), \vartheta(t), \dot z(t), \dot\vartheta(t)) = (0, \vartheta^* + \varepsilon t, 0, \varepsilon)$

$\downarrow$

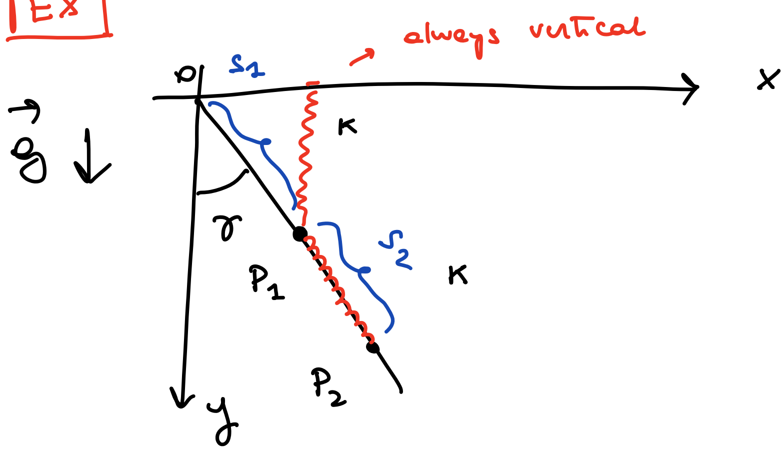
Sol. starting from $(0, \vartheta^*, 0, \varepsilon)$

$\downarrow$

unstable!

—x—x—

$\boxed{\text{EX}}$



P_1, m

P_2, m

—x—x—