Lesson 25 - 28/11/2022

- Conclusion of last exercise pevious lesson.
- Small oscillation.
- Ex 1
- Ex 2

• We study (un)stability of eq. $\underline{(\varphi,\varphi,0,0)}$

We take an initial datum:

$(\varphi,\varphi,\varepsilon,\varepsilon)$ $\quad\varepsilon$ small.

From explicit Lag. eqs, we conclude that

$$(\varphi(t),\vartheta(t),\dot{\varphi}(t),\dot{\vartheta}(t)) =$$
$$= (\varphi+\varepsilon t,\ \varphi+\varepsilon t,\ \varepsilon,\ \varepsilon)$$

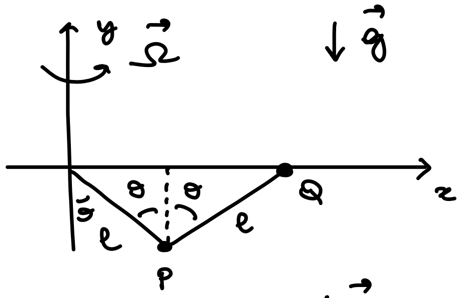
$\downarrow$

starting from $(\varphi,\varphi,\varepsilon,\varepsilon)$

$\Downarrow$

we conclude the unstability.

Ex 1

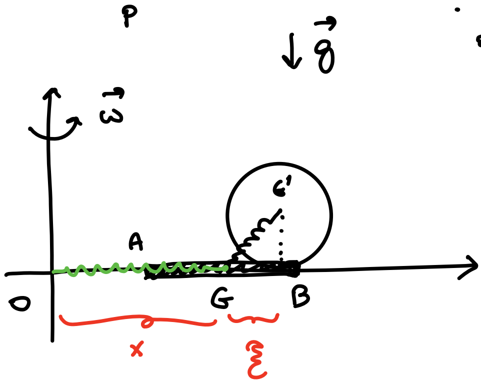


Q, P mass m

- Lagrangian L
- Eq. stability
- Bif. diagram of eq.
- Freq. of small oscillation around $\vartheta = 0$ (when stable).

Ex 2

IMPORTANT!



BAR : m

Disc : m, R purely rotating on the bar!!

SPRINGS : $k = 4m\omega^2$ as elastic constant.

LAGR. COO : $x = x_G$, $\xi = x_{G'} - x_G$.

→ Equilibria & their stability.

→ K and its kinetic matrix.

SMALL OSCILLATION

$L = K - V$

$x^* = (0,0) \in \mathbb{R}^{2n}$ stable eq. for a mechanical system:

$$L(q, \dot q) = \frac{1}{2} \sum_{h,k=1}^{m} a_{h,k}(q)\, \dot q_k \dot q_h - V(q).$$

We suppose that the stability of the equilibrium can be recognized by V, that is:

$$\nabla V(0) = 0 \quad \text{and} \quad (\text{Hess}\, V(0)\lambda, \lambda) > 0 \quad \forall \lambda \in \mathbb{R}^m.$$

Recall the normal form of L eqs:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_h} - \frac{\partial \mathcal{L}}{\partial q_h} = 0 \iff \dots \iff$$

$$\sum_{k=1}^{m} a_{hk}(q)\, \ddot q_k = \sum_{j,k=1}^{m} \left(\frac{1}{2}\frac{\partial a_{jk}}{\partial q_h}(q) - \frac{\partial a_{hk}}{\partial q_j}(q) \right) \dot q_j \dot q_k - \frac{\partial V}{\partial q_h}$$

$$= g_h(q, \dot q) - \frac{\partial V}{\partial q_h}(q)$$

$$\iff a(q)\, \ddot q = -\nabla V(q) + g(q, \dot q)$$

$\Leftrightarrow \ddot{q} = a^{-1}(q)\left[g(q,\dot{q})\right] - a^{-1}(q)\nabla V(q)$

$\Downarrow$ At first order

$$\begin{cases} \dot{q} = \xi \\ \dot{\xi} = a^{-1}(q)\left[\, g(q,\dot{q})\right] - a^{-1}(q)\nabla V(q) \end{cases}$$

We can linearize this (Lagrangian) vector field around the equilibrium : $(0,0) \in \mathbb{R}^{2n}$. We obtain:

$$\begin{cases} \dot{q} = \xi \\ \dot{\xi} = -a^{-1}(0)\,\text{Hess}\,V(0)\,q \end{cases}$$

$\Leftrightarrow$ $\boxed{a(0)\ddot{q} = -\text{Hess}(V(0))q}$ $\longrightarrow$ Linearization of Lagrange eqs for $L = K - V$ around $(0,0)$.

$\downarrow$

At second order

Remark

Previous eq. is the Lagrange eq. for the Lagrangian :

$$L^{\circ}(q,\dot{q}) = \frac{1}{2}\sum_{h,k=1}^{n} a_{h,k}(0)\,\dot{q}_h\dot{q}_k - \frac{1}{2}\sum_{h,k=1}^{n}\frac{\partial^2 V(0)}{\partial q_h \partial q_k}q_h q_k$$

In fact :

$$\frac{d}{dt}\frac{\partial \mathcal{L}^{\circ}}{\partial \dot{q}_h} - \frac{\partial \mathcal{L}^{\circ}}{\partial q_h} = 0 \Leftrightarrow \frac{d}{dt}\left(a(0)\dot{q}\right) + \text{Hess}\,V(0)q = 0$$

$$\Leftrightarrow a(0)\ddot{q} = -\text{Hess}\,V(0)\,q$$

Now, we use 3 changes of variables in order to simplify pevious eqs of second order giving the linearization around $(0,0)$.

Observe that :

$$K^\circ = \frac{1}{2}\sum_{h,k=1}^{m} a_{h,k}(0)\,\dot q_h \dot q_k$$ symmetric, + def.

$$V^\circ = \frac{1}{2}\sum_{h,k=1}^{m} \frac{\partial^2 V}{\partial q_h \partial q_k}(0)\, q_h q_k$$ symmetric, + def (by hypothesis).

1) $q \mapsto q'$ in order to diagonalize $a_{h,k}(0)$.

(in fact, $a_{h,k}$ is symm. and + def $\Rightarrow$ is orthogonally diagonalizable : $\exists R$ $q' = Rq$ s.t. $a_{h,k}(0)$ becomes diagonal). In these coordinates:

$$\begin{cases} K^\circ = \frac{1}{2}\sum_{h=1}^{m} a_h (\dot q'_h)^2 & a_h > 0 \\ V^\circ = \frac{1}{2}\sum_{h,k=1}^{m} V'_{h,k}\, q'_h q'_k \end{cases}$$

2) $q''_h = \sqrt{a_h}\, q'_h$

$$\begin{cases} K^\circ = \frac{1}{2}\sum_{h=1}^{m} (\dot q''_h)^2 \\ V^\circ = \frac{1}{2}\sum_{h,k=1}^{m} \frac{V'_{h,k}}{\sqrt{a_h a_k}}\, q''_h q''_k \end{cases}$$

3) $q'' \longmapsto x = \tilde R q''$ in order to diagonalize $\frac{V'_{h,k}}{\sqrt{a_h a_k}}$

$$\begin{cases} K^\circ = \frac{1}{2}\sum_{h=1}^{m} (\dot x_h)^2 \\ V^\circ = \frac{1}{2}\sum_{h=1}^{m} \omega_h^2 x_h^2 \end{cases}$$

In these new variables, $L^\circ(x, \dot x) = K^\circ - V^\circ =$

$$= \frac{1}{2}\sum_{h=1}^{m} (\dot x_h)^2 - \frac{1}{2}\sum_{h=1}^{m} \omega_h^2 x_h^2$$

and Lagrange eqs:

$$\boxed{\ddot x_h = -\omega_h^2 x_h} \longrightarrow$$ m equations of harmonic oscillators (springs) each with period $T_h = 2\pi/\omega_h$

ω_n ?! CHARACTERISTIC FREQUENCIES ?

We recall that we operated 3 changes of coordinates such that $q \mapsto x = \underset{\downarrow}{S}q \iff q = S^{-1}x$

composition of 1, 2, 3.

In particular:

S diagonalizes $a(0)$ into $\mathbb{1}$: $S^{-T}a(0)S^{-1} = \mathbb{1}$

S diagonalizes $\text{Hess}\,V(0)$ into $\underbrace{\begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{pmatrix}}_{= \Omega^2}$:

$S^{-T}\,\text{Hess}\,V(0)\,S^{-1} = \Omega^2$

Now:

ω is a characteristic frequency iff ω^2 is an eigenvalue of Ω^2. Therefore:

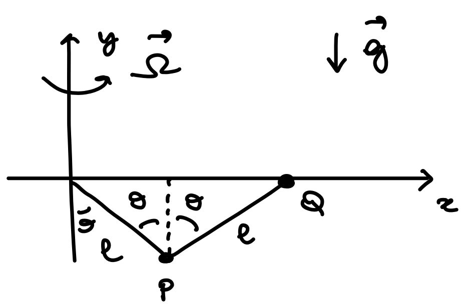
$$0 = \det(\Omega^2 - \omega^2 \mathbb{1}) \iff$$

$$0 = \det\left(S^{-T}\,\text{Hess}\,V(0)\,S^{-1} - \omega^2 S^{-T} a(0) S^{-1}\right) \iff$$

$$\boxed{0 = \det\left(\text{Hess}\,V(0) - \omega^2 a(0)\right)}$$

eq. for characteristic frequencies

Ex 1



Q, P mass m

- Lagrangian L
- Eq. stability
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- Freq. of small oscillation around $\theta = 0$ (when stable).

$$\vec{OQ} = (\ell\sin\theta + \ell\sin\theta, 0) \Rightarrow$$

$$\vec{v}_Q = (2\ell\dot\theta\cos\theta, 0)$$

$$\Rightarrow K_Q = \frac{1}{2} m (2\ell)^2 \cos^2\theta\, \dot\theta^2$$

$\vec{OP} = (\ell \sin\vartheta, -\ell\cos\vartheta) \Rightarrow$

$\vec{v}_P = (\ell\dot\vartheta\cos\vartheta, \ell\dot\vartheta\sin\vartheta)$

$\Rightarrow K_P = \frac{1}{2} m \ell^2 \dot\vartheta^2$

$K_{TOT} = \frac{1}{2} m \ell^2 (4\cos^2\vartheta + 1)\dot\vartheta^2$

$V_{gr} = -mg\ell\cos\vartheta$ + consts.

$V_{cf} = -\frac{1}{2} m\Omega^2 [(\ell\sin\vartheta)^2 + (2\ell\sin\vartheta)^2]$
$= -\frac{5}{2} m\Omega^2\ell^2\sin^2\vartheta$

$L = K_{TOT} - V_{gr} - V_{cf}$

Equilibria

$V'(\vartheta) = mg\ell\sin\vartheta - 5m\Omega^2\ell^2\sin\vartheta\cos\vartheta$
$= m\ell\sin\vartheta(g - 5\Omega^2\ell\cos\vartheta) = 0$

a) $\vartheta = 0, \pi$

b) $\cos\vartheta = \frac{g}{5\Omega^2\ell}$ if $\frac{g}{5\Omega^2\ell} < 1$ In such a case:

$\vartheta^* = \arccos\left(g/5\Omega^2\ell\right)$ and $-\vartheta^*$.

Stability ?!

$V''(\vartheta) = mg\ell\cos\vartheta - 5m\Omega^2\ell^2\cos^2\vartheta + 5m\Omega^2\ell^2\sin^2\vartheta$

- $V''(0) = mg\ell - 5m\Omega^2\ell^2$ stable if

$mg\ell - 5m\Omega^2\ell^2 > 0 \iff \frac{g}{5\Omega^2\ell} > 1$

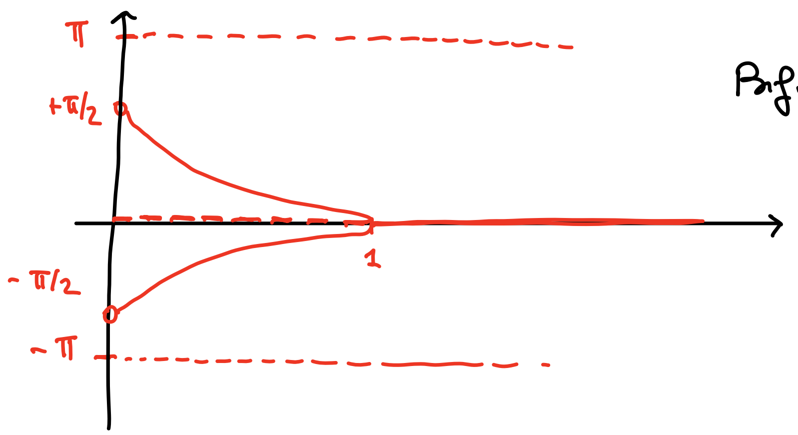
unstable otherwise.

- $V''(\pi) = -mge - 5m\Omega^2 e^2 < 0$ always unstable.
- $V''(\theta^*) = V''(-\theta^*) = \dots =$

$$\cancel{\frac{mg^2}{5\Omega^2}} - \cancel{\frac{mg^2}{5\Omega^2}} + \underbrace{5m\Omega^2 e^2}_{>0} \left(\frac{25\Omega^4 e^2 - g^2}{\underbrace{25\Omega^4 e^2}_{>0}} \right) > 0$$

iff $25\Omega^4 e^2 - g^2 > 0$

$\Leftrightarrow \dfrac{g}{5\Omega^2 e} < 1 \to$ always stable



Bif. diagram.

Suppose $\lambda = g/5\Omega^2 e \geq 1 \Rightarrow 0$ is stable.

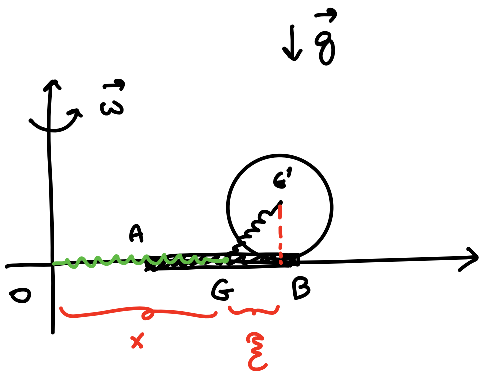
ω = charact. frequency:

$$\omega^2 = \frac{V''(0)}{5me^2} =$$

$$= \frac{mge - 5m\Omega^2 e^2}{5me^2} = \frac{g}{5e} - \Omega^2$$

Ex 2

IMPORTANT!



BAR : m

Disc : m, R purely rotating. on the bar !!

SPRINGS : $h = 4m\omega^2$ as elastic constant.

LAGR. COO : $x = x_G$, $\xi = x_{G'} - x_G$.

→ Equilibria & their stability.

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The Coriolis force $\vec{F} = -2m\,\vec{\omega} \wedge \dot{\vec{q}}$ has work $\equiv 0$.
because $\vec{\omega}$, $\dot{\vec{q}}$ and $\delta\vec{q}$ are all on the plane Oxy.

$$V = V^{cf} + V^{el}.$$

$$V^{el} = V^{el}(x, \xi) = \frac{h}{2}\left(x^2 + \xi^2\right) + \text{const.}$$

$$V^{cf} = V^{cf}(x, \xi) = -\frac{\omega^2}{2} m \left(x^2 + (x+\xi)^2\right) + \text{consts}$$

↓

Huygens - Steiner theo.

Equilibria :

$$\begin{cases} 0 = \dfrac{\partial V}{\partial x} = (h - \omega^2 m)\,x - m\omega^2 (x + \xi) \\[2ex] 0 = \dfrac{\partial V}{\partial \xi} = h\,\xi - \omega^2 m (x + \xi) \end{cases}$$

$\underline{h = 4m\omega^2}$

$$\begin{cases} 0 = 2m\omega^2 x - m\omega^2 \xi \\[2ex] 0 = -m\omega^2 x + 3m\omega^2 \xi \end{cases} \implies \exists!\ \text{equilibrium } (0,0) \in \mathbb{R}^2.$$

$$\mathrm{Hess}\, V(x,\xi) = m\omega^2 \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

positive def. since $\underline{2>0}$ and $\underline{\det = 6-1 = 5 > 0}$

$\Rightarrow (0,0)$ is stable!

<u>Kinetic energy.</u>

We need to introduce an angle θ : $R\theta = \xi$ for the kinetic energy for the disc.
For the bar, it's sufficient to consider $\dot{x}$.
Therefore:

$$\dot{\xi}^2/\cancel{R^2}$$

$$K(x,\xi,\dot{x},\dot{\xi}) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m(\dot{x}+\dot{\xi})^2 + \frac{1}{2}\frac{m\cancel{R^2}}{2}\dot{\theta}^2$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m(\dot{x}^2+\dot{\xi}^2+2\dot{x}\dot{\xi}) + \frac{1}{2}\frac{m}{2}\dot{\xi}^2$$

$$= \frac{1}{2} m(2)\dot{x}^2 + \frac{1}{2} m\left(\frac{3}{2}\right)\dot{\xi}^2 + \frac{1}{2} m\, 2\dot{x}\dot{\xi}$$

$$\Rightarrow Q(x,\xi) = m\begin{pmatrix} 2 & 1 \\ 1 & 3/2 \end{pmatrix}$$

$$K = \frac{1}{2}\begin{pmatrix}\dot{x} \\ \dot{\xi}\end{pmatrix}^T \begin{pmatrix} 2m & m \\ m & \frac{3}{2}m \end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{\xi}\end{pmatrix}$$

—x—x—