Theorem $f: I \to \mathbb{R}$ (I is an open interval), f differentiable on I.

f is convex $\Longleftrightarrow$ $f': I \to \mathbb{R}$ is increasing

Proof "$\Longrightarrow$" $\boxed{x<y}$

By convexity, we have

$$f(y) \geq f(x) + f'(x)(y-x)$$

$$f(x) \geq f(y) + f'(y)(x-y)$$

$$\frac{f(x)-f(y)}{x-y} = \frac{f(y)-f(x)}{y-x} \geq f'(x)$$

$$f(x) - f(y) \geq f'(y)(x-y)$$

$$\frac{f(x)-f(y)}{x-y} \leq f'(y)$$

$$f'(x) \le \frac{f(x)-f(y)}{x-y} \le f'(y)$$

$$\parallel$$

$$\frac{f(y)-f(x)}{y-x}$$

$$\Downarrow$$

$x<y$ $\quad f'(x) \le f'(y)$

" $\Leftarrow$ " Assume that f' is increasing

$x<y$ $\quad f'(x) \le f'(y)$

By Lagr. Th.

$\exists\, \xi \in]x,y[$ $\quad \frac{f(y)-f(x)}{y-x} = f'(\xi)$

$$f'(x) \le f'(\xi) = \frac{f(y)-f(x)}{y-x} \le f'(y)$$

$$f(y)-f(x) \le f'(y)(y-x)$$

$$f(x)-f(y) \ge f'(y)(x-y)$$

$$\boxed{f(x) \ge f(y) + f'(y)(x-y)}$$

$$(y-x)f'(x) \le \frac{f(y)-f(x)}{\cancel{y-x}}$$

$$\boxed{f(y) \ge f(x) + f'(x)(y-x)}$$

q.e.d.

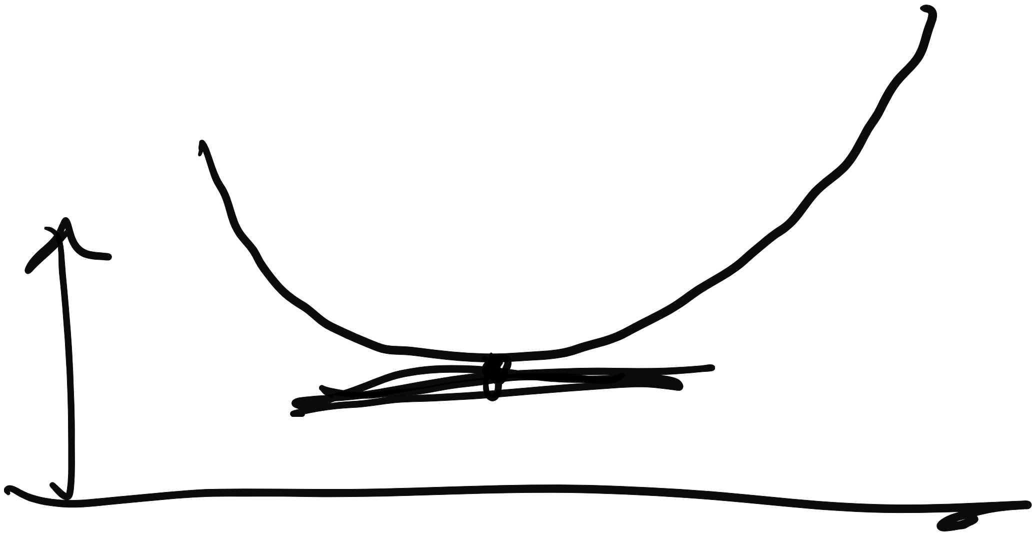
Corollary. $f : I \to \mathbb{R}$ has second derivative at each $x \in I$

f convex $\Leftrightarrow f''(x) \geq 0$
$\forall x \in I$

Theorem: $f : I \to \mathbb{R}$ is convex and x_0 interior to I is s.t. $f'(x_0) = 0$

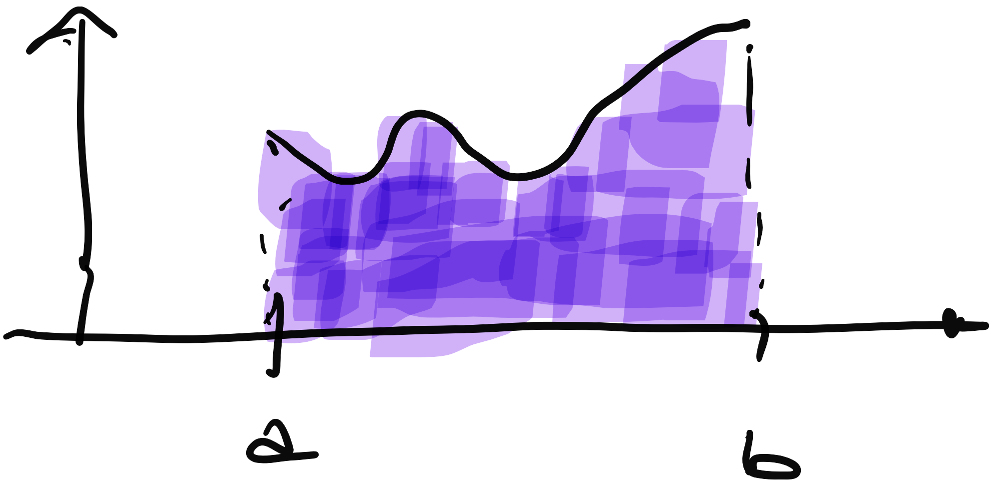
Then x_0 is a (absolute) minimum point

Proof $\boxed{f(x)} \geq f(x_0) + \underbrace{f'(x_0)(x - x_0)}_{}$
$\forall x \in I$ $\qquad = \boxed{f(x_0)}$

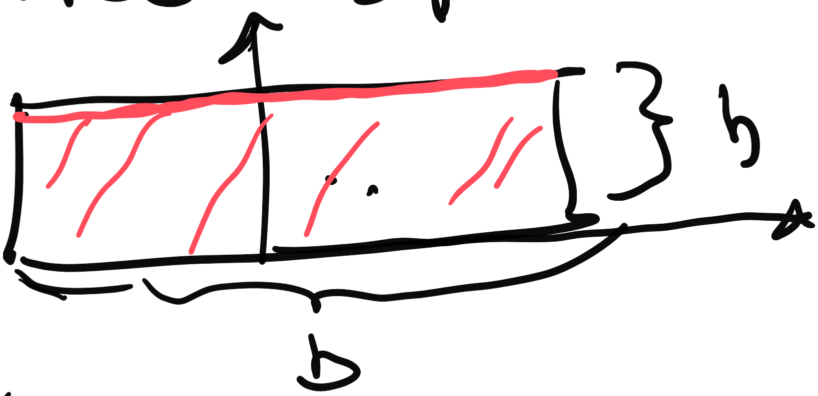


TWO PROBLEMS:

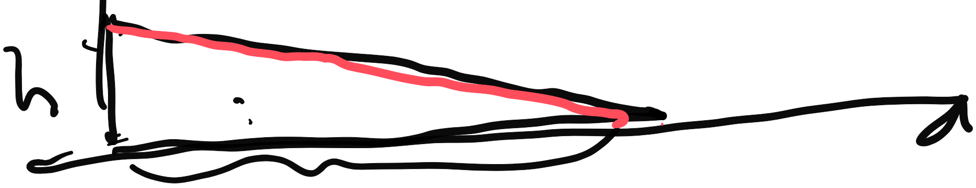
I) Given a function $f : [a,b] \to \mathbb{R}$, $f(x) \geq 0$



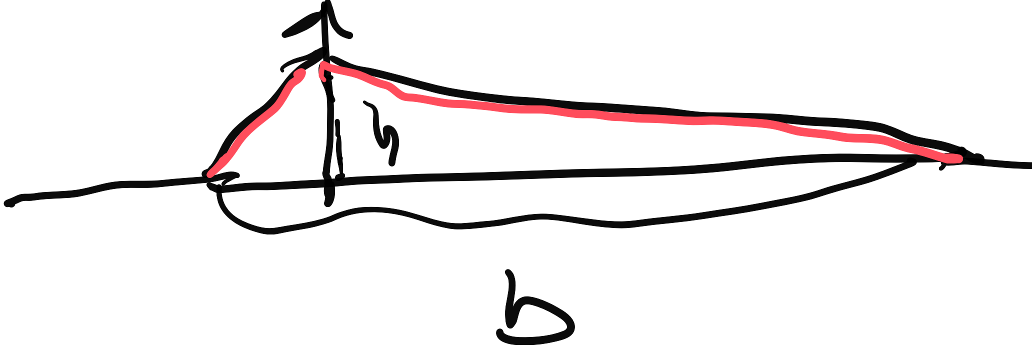
We know what an "Area" of a rectangle is



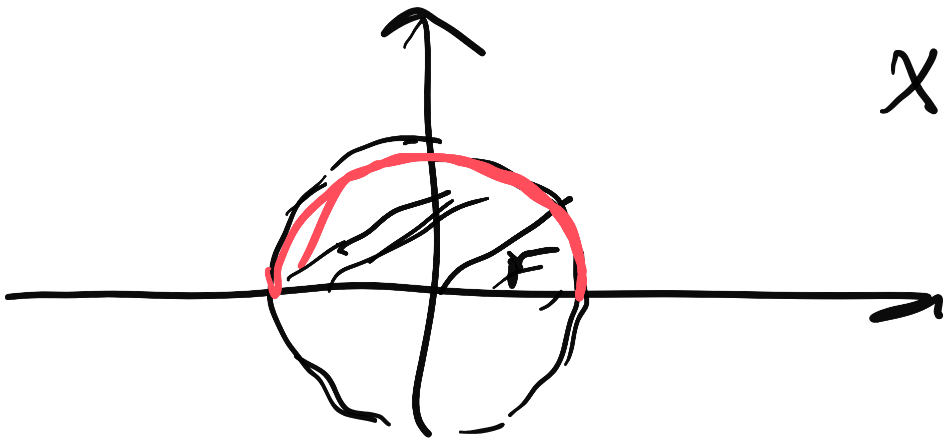
$$\text{Area} = b \cdot h$$



$$\text{Area} = \frac{1}{2} h \cdot b$$



Area $= \frac{1}{2} h \cdot b$



$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2}$

Area $= \pi r^2$

II) Find the "inverse operation of differentiation"

i.e.

$F : I \longrightarrow \mathbb{R}$ F is differentiab

$f = F' : I \longrightarrow \mathbb{R}$

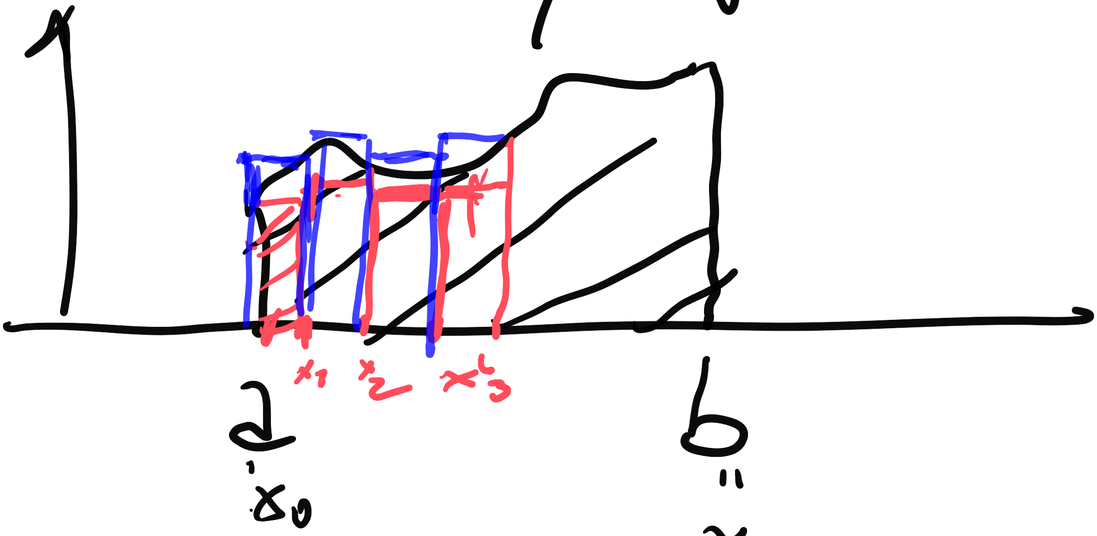
Given $f: I \to \mathbb{R}$

can we construct

$F: I \to \mathbb{R}$

s.t. $f = F'$?

Problem I: define Area of the "trapezoid" defined by $f: [a,b] \to \mathbb{R}$



$$\Pi = \{a = x_0 < x_1 < x_2 \cdots < x_n = b\}$$

$$\underline{S}(\Pi) = (x_1 - x_0) m_0 + (x_2 - x_0) m_1 + \cdots (x_n - x_{n-1}) m_{n-1}$$

: inferior sum

$$m_0 = \inf_{x \in [x_0, x_1]} f(x) \qquad m_1 = \inf_{x \in [x_1, x_2]} f(x)$$

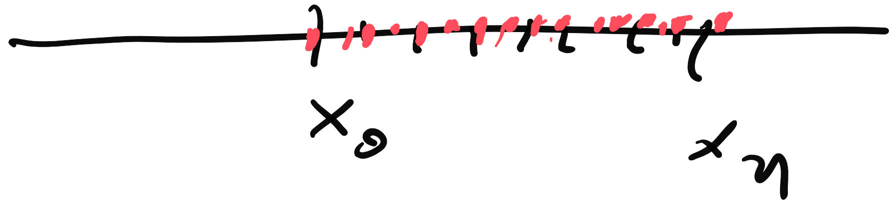
$$m_k = \inf_{x \in [x_k, x_{k+1}]} f(x) \qquad k = 0, 1, \dots, n-1$$

$$\bar{S}(\pi) = (x_1 - x_0) M_0 + (x_2 - x_1) M_1 + \dots + (x_n - x_{n-1}) M_{n-1}$$

$$M_k = \sup_{x \in [x_k, x_{k+1}]} (f(x))$$

superior sum

$$\underline{S}(\pi) \leq \bar{S}(\pi)$$



$$\tilde{\Pi}_2 \supseteq \tilde{\Pi}_1$$

$$\tilde{\Pi}_1 \subseteq \tilde{\Pi}_2$$

$$\underline{S}(\tilde{\Pi}_1) \le \underline{S}(\tilde{\Pi}_2)$$

$$\overline{S}(\tilde{\Pi}_1) \geq \overline{S}(\tilde{\Pi}_2)$$

$$\underline{A}(f) = \sup_{\tilde{\Pi} \text{ subdivisão}} \underline{S}(\tilde{\Pi})$$

inferior área

$$\overline{A}(f) = \inf_{\tilde{\Pi} \text{ subdivisão}} \overline{S}(\tilde{\Pi})$$

superior área.

Definition: $f : [a,b] \to \mathbb{R}$

if $\underline{A}(f) = \overline{A}(f)$

I say that

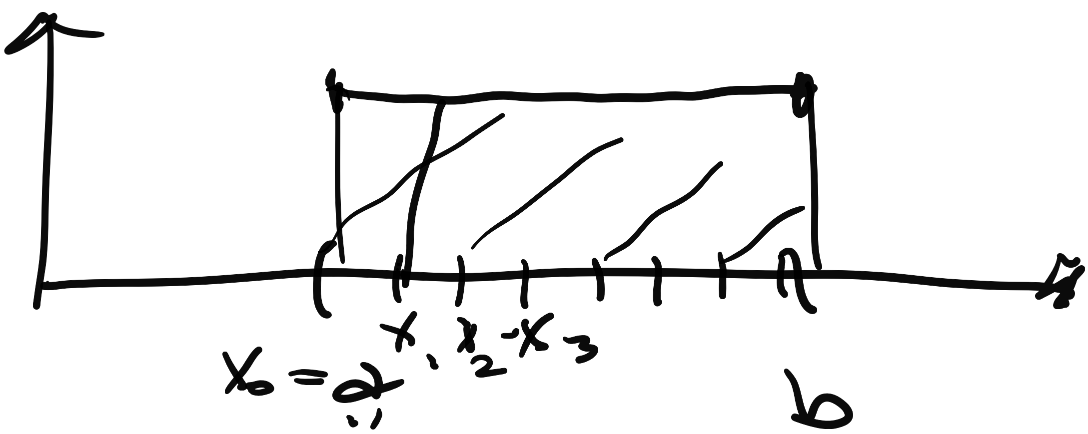
$$A(f) := \underline{A}(f) = \overline{A}(f)$$

is the

AREA OF THE f-trapezoid

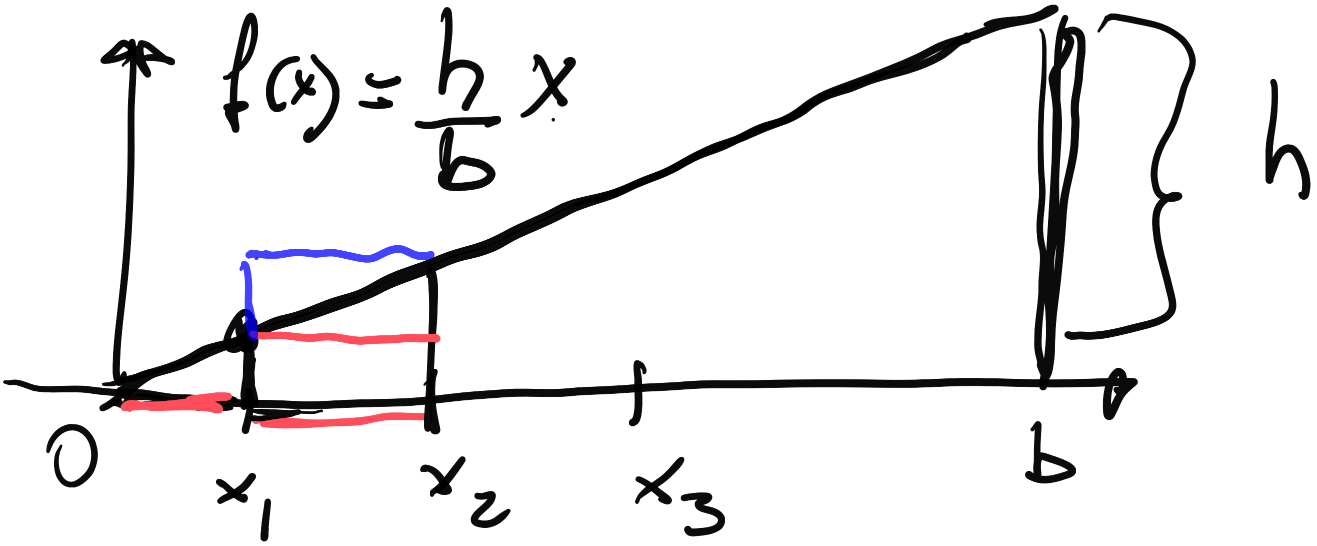
<u>Examples</u>.

$f(x) = c > 0$



$$\underline{S}(\pi) = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, m_k =$$

$$= c \underbrace{\sum_{k=0}^{n-1} (x_{k+1} - x_k)} = c\,(b-a)$$



$$\underline{S}(\Pi) = \cancel{x_1} \cdot 0 + (x_2 - x_1)\frac{h}{b} x_1 +$$

$$+ (x_{k+1} - x_k)\frac{h}{b} x_k$$

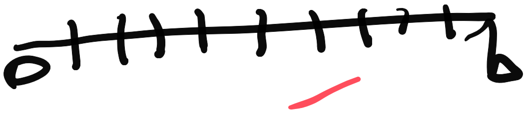
$$+ (x_n - x_{n-1}) \cdot \frac{h}{b} x_{n-1}$$

$$= \sum_{k=0}^{n-1} \overbrace{(x_{k+1} - x_k)}^{\frac{b}{n}} \frac{h}{b} x_k =$$

$$\sum_{0}^{n-1} \frac{b}{n} \cdot \frac{h}{\cancel{b}} \, \frac{k \cancel{b}}{n} =$$

Choose the special subdivision Π_n

$$\Pi_n = \left\{ x_1 = \frac{b}{n} \quad x_2 = \frac{2b}{n} \quad \dots\dots \quad x_{n-1} = \frac{(n-1)b}{n} \quad x_n = b \right\}$$



$$\frac{bh}{n^2} \sum_{k=0}^{n-1} k = \frac{bh}{n^2} \frac{(n-1)n}{2} =$$

$$= \boxed{\frac{(n-1)n}{n^2}} \cdot \boxed{\frac{bh}{2}} \xrightarrow{n \to +\infty} \frac{bh}{2}$$

$$\frac{n^2 - n}{n^2} = 1 - \frac{1}{n} \to 1$$

$$\overline{S}(\Pi_n) = \sum_{k=0}^{n-1} \underline{(x_{k+1} - x_k)} \frac{h}{b} x_{k+1}$$

$$= \sum_{k=0}^{n-1} \frac{\cancel{b}}{n} \frac{h}{\cancel{b}} \cdot (k+1) \frac{b}{n} =$$

$$\frac{hb}{n^2}\sum_{k=0}^{n-1} k+1 = \frac{bh}{n^2}\sum_{\hat{k}=1}^{n}\hat{k} =$$

$$\boxed{\frac{hb}{n^2}\,\frac{n(n+1)}{2}} \xrightarrow{n\to\infty} \frac{bh}{2}$$

Find the area of the trapezoid