Definition $: D \longrightarrow \mathbb{R}$

$x_0 \in \text{int}(D)$, f differentiable on a neighborhood of x_0.

$$f''(x_0) := (f')'(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

SECOND DERIVATIVE

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notation $\boxed{f''(x_0)}$ or $\boxed{f^{(2)}(x_0)}$ or $\boxed{\frac{d^2 f}{dx^2}(x_0)}$

Examples

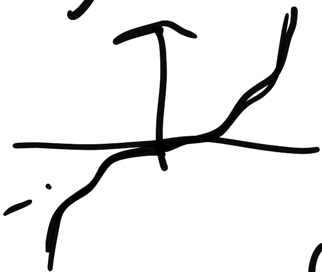
- $f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$
- $f(x) = \log x$ $\quad f'(x) = \frac{1}{x}$ $\quad f''(x) = -\frac{1}{x^2}$

In general, for every $n \in \mathbb{N}$, the n-th derivative as

$$f^{(n)}(x_0) = \left(f^{(n-1)}\right)'(x_0)$$

$(\sin)^{(2)}(x) = -\sin(x)$ $\quad$ $(\sin)^{(3)}(x) = -\cos(x)$

$\sin^{(4)}(x) = \sin(x)$



$f(x) = x|x|$

$x \neq 0$ $\quad f'(x) = \begin{cases} 2x & \text{if } x>0 \\ -2x & \text{if } x<0 \end{cases}$

$$f'(0) = \lim_{x\to 0} \frac{\cancel{x}|x|}{\cancel{x}} = 0$$

$$f''_+(0) = \lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x} = \lim \frac{2x}{x} = 2$$

$$f''_-(0) = \lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0^-} \frac{-2x}{x} = -2$$

the second derivative does not exist in $x=0$

PROBLEM

We know that the existence of $f'(x_0)$ is equivalent to

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

DOES A BETTER APPROXIMATION, SAY

$$f(x) \overset{?}{=} c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + o\left((x-x_0)^2\right)$$

EXIST?

EVEN MORE GENERALLY:

$$f(x) \overset{?}{=} c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots + c_n(x-x_0)^n + o\left((x-x_0)^n\right)$$

The case of $f(x) = p(x)$ polynomial!

$$p(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots + c_n(x-x_0)^n$$

$$p'(x) = c_1 + 2c_2(x-x_0) + \dots + nc_n(x-x_0)^{n-1}\Big|_{x=x_0}$$

$$p'(x_0) = \boxed{c_1}$$

$$p''(x) = 2c_2 + 6c_3(x-x_0) + \dots + n(n-1)(x-x_0)^{n-2}\Big|_{x=x_0}$$

$$\Rightarrow p''(x_0) = 2c_2 \Leftrightarrow c_2 = \frac{p''(x_0)}{2!}, \quad 2! = 2$$

$$p'''(x_0) = 2\cdot 3\,c_3 \Leftrightarrow c_3 = \frac{f'''(x_0)}{3!}$$

$$p(x) = p(x_0) + p'(x_0)(x-x_0) + \frac{p''(x_0)}{2!}(x-x_0)^2 + \dots + p^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

We would like, for generic f

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o((x-x_0)^n) \quad (*)$$

YES: If f is differentiable n times then $(*)$ holds true. (see Theorem 1 below)

$$\sin x = x - \frac{x^3}{6} + \boxed{\frac{x^5}{120}} + o(x^5)$$

This was given... let us check it.

From $(*)$:

$x_0 = 0$

$$\sin x = f'(0)x + \frac{f''(0)}{2}x^2 + \boxed{\frac{f'''(0)}{6}x^3} + \cancel{\frac{f^{(4)}(0)}{24}x^4} + \frac{f^{(5)}(0)}{120}x^5 + o(x^5)$$

$f'(0) = (\sin)'(0) = \cos 0 = 1$

$f''(0) = (\sin')'(0) = -\sin 0 = 0$

$f'''(0) = (\sin)''(0) = -\cos(0) = -1$

$f^{IV}(0) = 0$

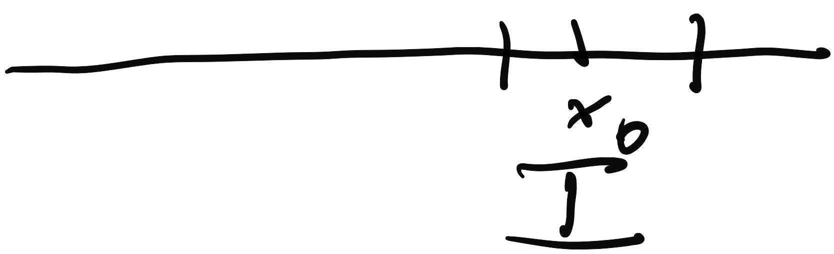
$f^{(V)}(0) = \cos(0) = 1$

Theorem 1: Consider $f : I \longrightarrow \mathbb{R}$ with I neighbourhood of x_0. Assume

- f is $(n-1)$ times differentiable
- $f^{(n)}(x_0)$ exists.

Then

$$(*)\quad f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Taylor polynomial of degree } n} + o\left((x-x_0)^n\right)$$



$$\boxed{o\left((x-x_0)^n\right) = 10^9 (x-x_0)^{n+1}}$$

Proof $(n=2)$

we wish to prove

$$f(x) \stackrel{?}{=} f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + o\left((x-x_0)^2\right)$$

$$\underbrace{f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{f''(x_0)}{2}(x-x_0)^2}_{} \stackrel{?}{=} o\left((x-x_0)^2\right)$$

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{f''(x_0)}{2}(x-x_0)^2}{(x-x_0)^2} \stackrel{H.}{=}$$

$$= \lim_{x\to x_0} \frac{f'(x) - f'(x_0) - 2\frac{f''(x_0)}{2}(x-x_0)}{2(x-x_0)} =$$

$$= \lim_{x\to x_0} \left(\frac{f'(x) - f'(x_0)}{2(x-x_0)} \right) - \frac{f''(x_0)}{2} =$$

$$= \frac{1}{2} f''(x_0) - \frac{1}{2} f''(x_0) = 0$$

Application

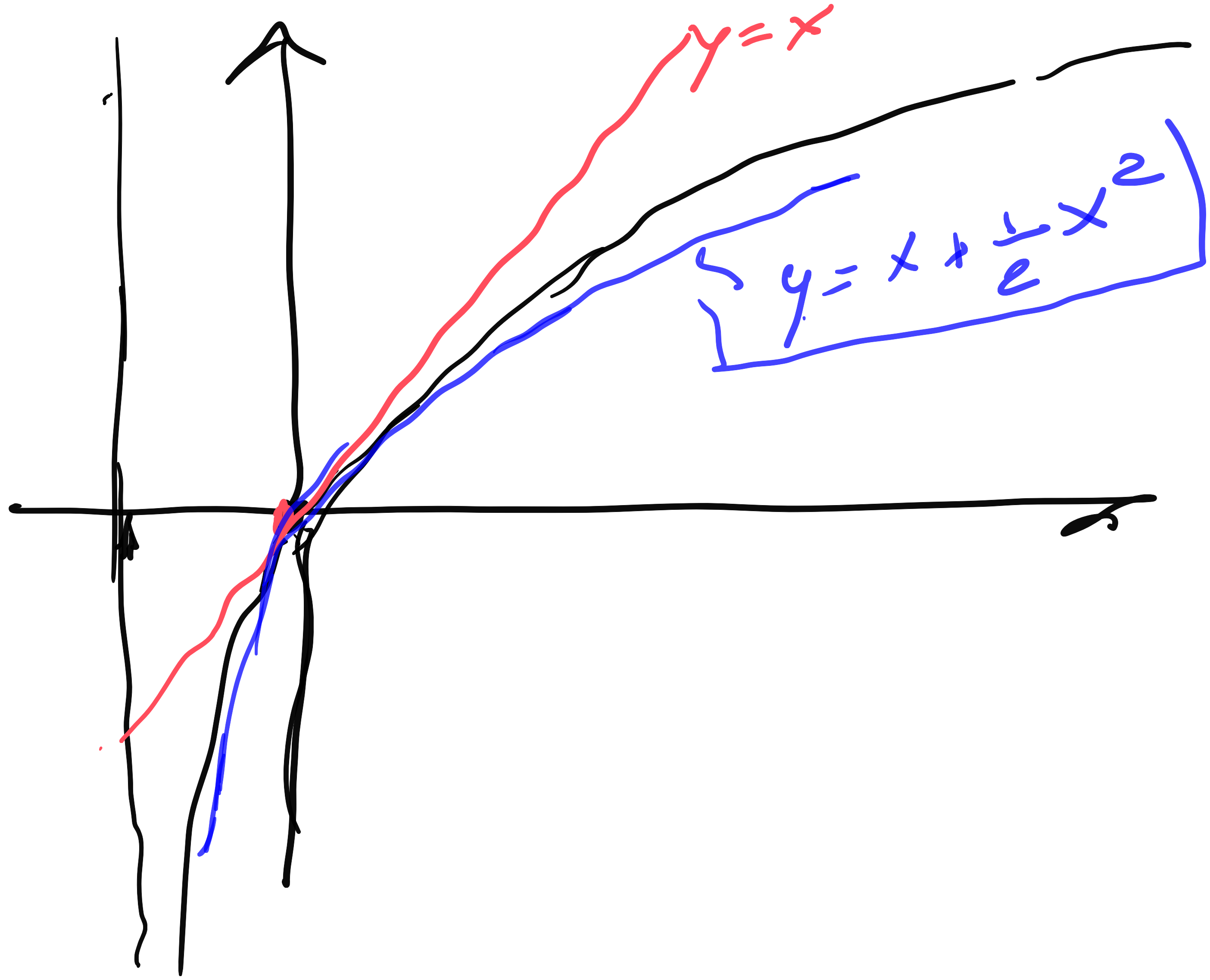
$$\log(1+x) = x + o(x)$$

indeed, Apply Taylor with $x_0 = 0$, to $f(x) = \log(1+x)$

$$\underline{\log(1+x)} = \log(1) + \frac{1}{(1+x)}\Big|_{x=0} x + o(x) = \boxed{x + o(x)}$$

$$\log(1+x) = \log(1) + \frac{1}{(1+x)}\Big|_{x=0} x - \frac{1}{2}\frac{1}{(1+x)^2}\Big|_{x=0} x^2 + o(x^2)$$

$$= \log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$



Even better:

Theorem 2 $f: I \to \mathbb{R}$

I neighborhood of x_0,

- $f^{(n+1)}$ exists in I

Then there exists $\xi \in [x_0, x]$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Exercise:

Find $\sin\left(\frac{1}{10}\right)$ with a error $\underbrace{\text{less than } \boxed{\frac{1}{10.000}}}$

$x_0 = 0 \quad \left(x = \frac{1}{10}\right)$

$$\sin\left(\frac{1}{10}\right) = \frac{1}{10} - \frac{1}{6.1000} + \underbrace{\frac{\sin(\xi)}{4!} \cdot \frac{1}{10000}}_{\text{error}}$$

$\sin^{(4)}(x) = \sin x$

$$|\text{error}| \leq \frac{1}{24 \cdot 10.000} = \boxed{\frac{1}{240.000}}$$

$$\simeq \frac{600 - 1}{6.000} = \frac{599}{6.000}$$

CONVEXITY:

Fig 1

$f(x) \le f(y) + f'(y)(x-y)$

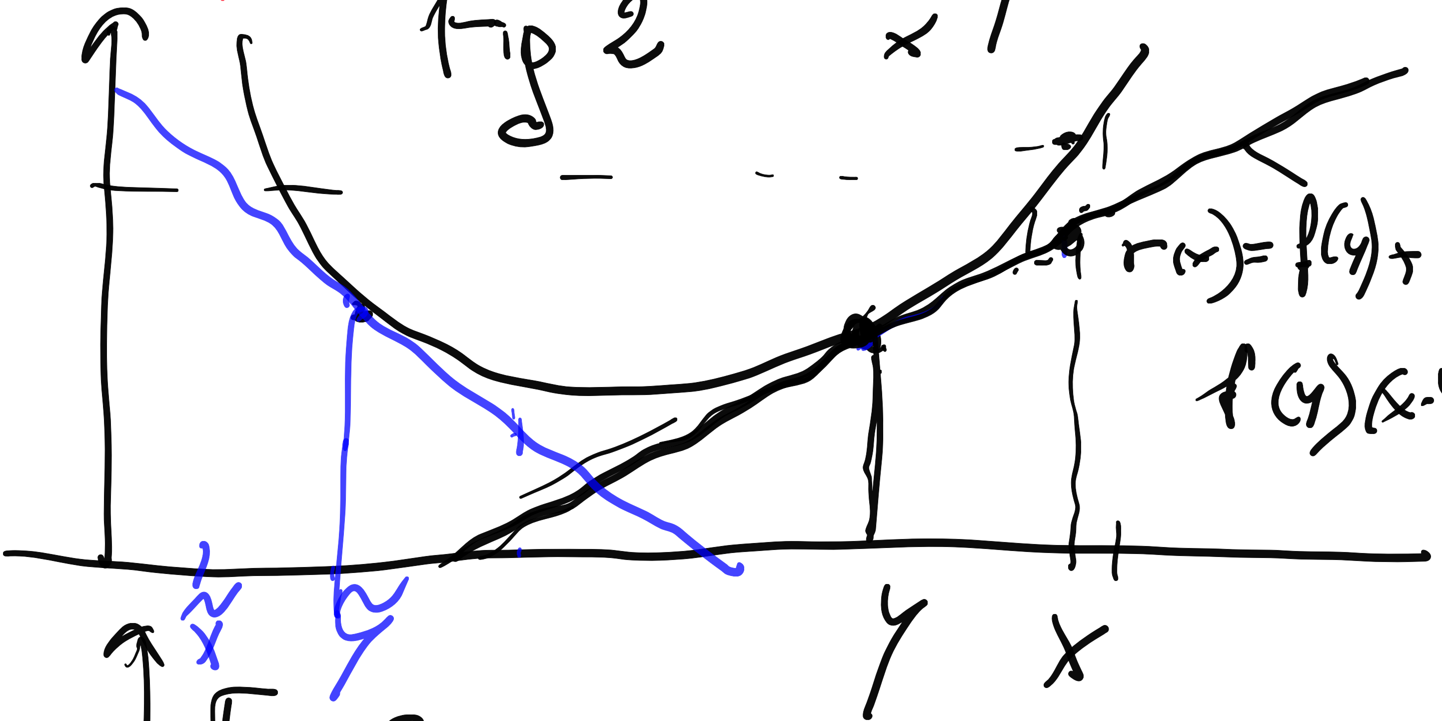


Fig 2

$r(x) = f(y) + f'(y)(x-y)$

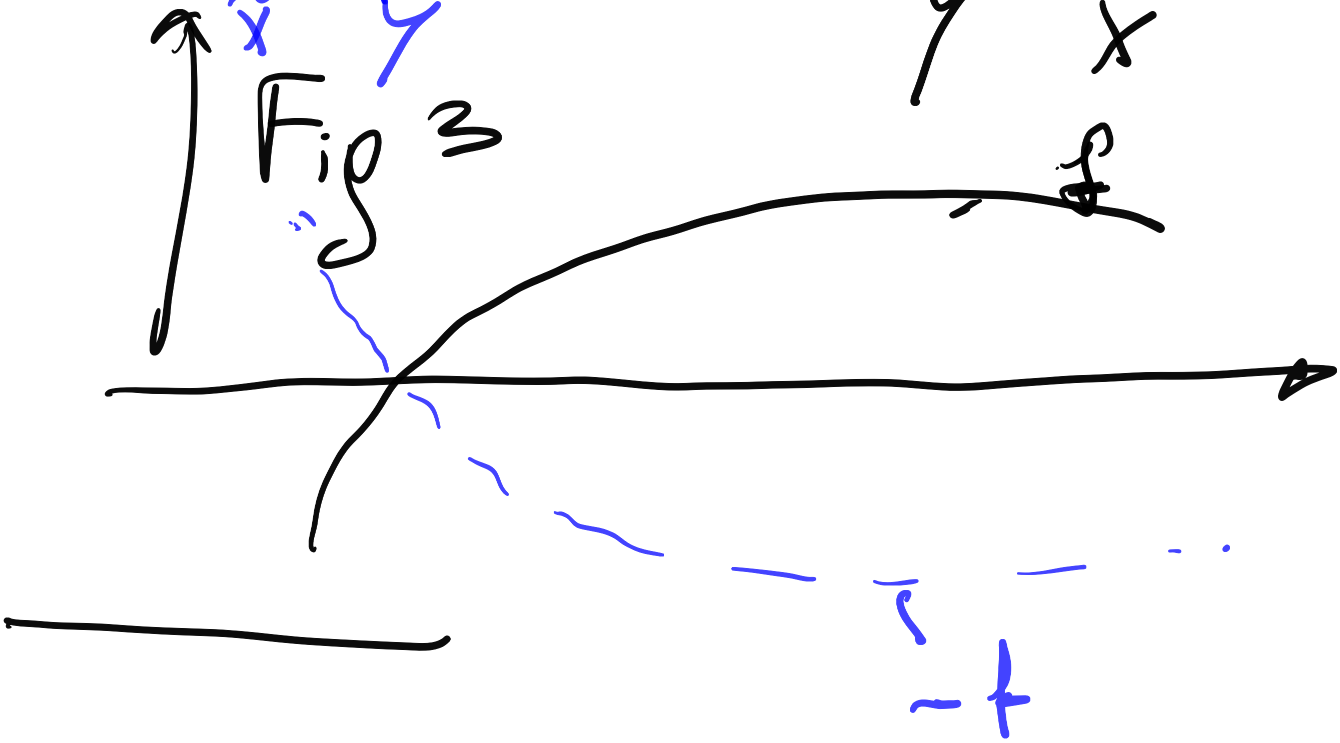


Fig 3

In Fig. 2 we hat

$$f(x) \geq f(y) + f'(y)(x-y)$$

So we say that

<u>Def</u> A differentiable function $f: I \to \mathbb{R}$ (I interval) is said <u>CONVEX</u> if $\forall x, y \in I$ we have

$$f(x) \geq f(y) + f'(y)(x-y)$$

Def: f is said CONCAVE if $-f$ is convex.

Theorem:

f is convex $\iff$ $f': I \to \mathbb{R}$ is increasing

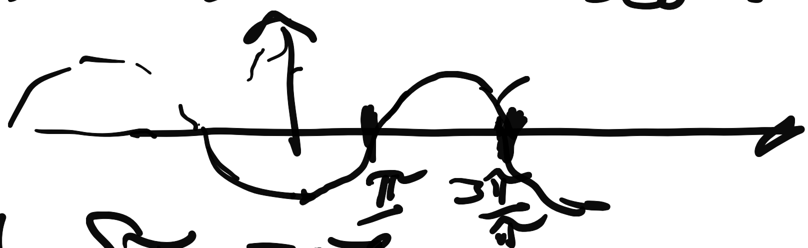
Corollary

Assume f is differentiable two times

f is convex $\iff$ $f''(x) \geq 0$ $\forall x \in I$.

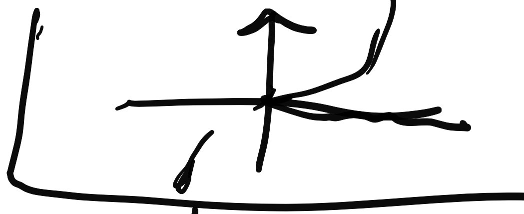
Find an interval $[a, b] \subseteq \mathbb{R}$ where $f(x) = \cos x$ is convex

$f''(x) = -\cos x \geq 0$ $\quad -\cos x$



Solution: $[a, b] = [\frac{\pi}{2}, \frac{3\pi}{2}]$

Exercise

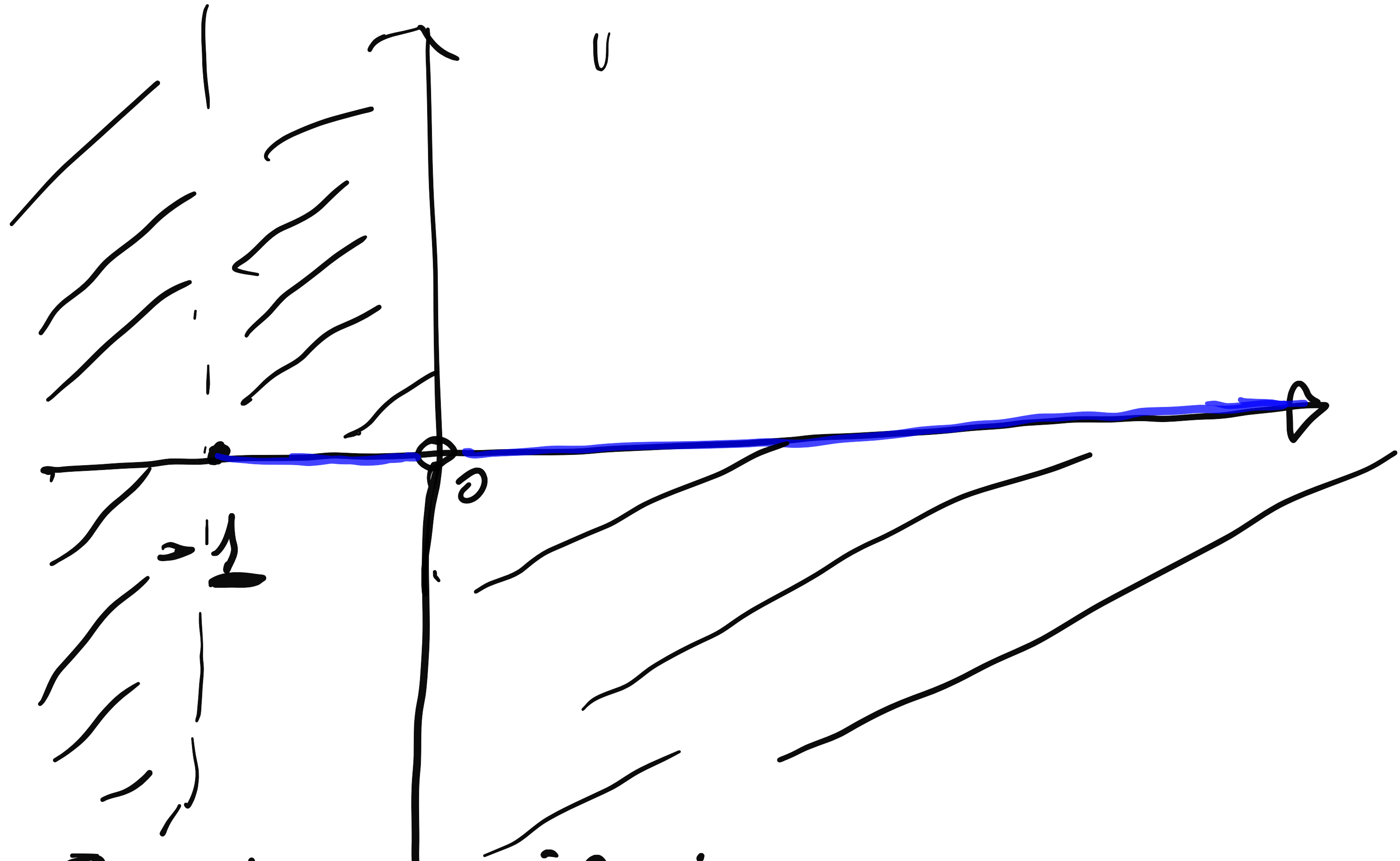


Study domain, sign, limits, asintotes, continuity, differentiability, f', limits of f'.

$$f(x) = \frac{(x+1)^{\frac{1}{3}}}{\left(\log(x+1)\right)^3}$$

Maximal domain:

$$D = \left\{ x\in\mathbb{R} \;\middle|\; \begin{array}{l}(x+1)>0\\ x+1\neq 1\end{array}\right\} = \;]-1,+\infty[\setminus \{0\}$$



Study of sign:

$$f(x) \geq 0 \qquad \frac{(x+1)^{\frac{1}{3}}}{\log^3(x+1)} \geq 0$$

$$\begin{cases} (x+1)^{\frac{1}{3}} \geq 0 \\ \log^3(x+1) > 0 \end{cases} \cup \begin{cases} (x+1)^{\frac{1}{3}} \leq 0 \\ \log^3(x+1) < 0 \end{cases}$$

$$\boxed{\begin{cases} x \geq -1 \\ x+1 > 1 \end{cases}} \cup \cancel{\begin{cases} x \leq -1 \\ x+1 < 1 \end{cases}}$$

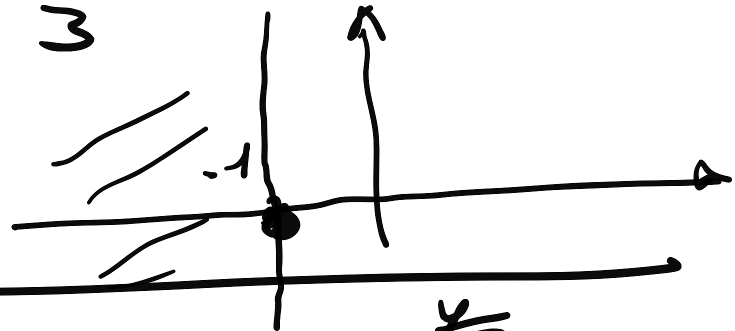
$f(x) \geq 0 \iff x > 0$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{(x+1)^{\frac{1}{3}}}{(\log(x+1))^3} =$

$y = \log(x+1)$

$(x+1) = e^y$

$\lim_{y \to -\infty} \frac{e^{y/3}}{y^3} = 0$



$\lim_{x \to +\infty} \frac{(x+1)^{\frac{1}{3}}}{\log(x+1)^3} = \lim_{y \to +\infty} \frac{e^{y/3}}{y^3} = +\infty$

$\lim_{x \to 0^-} \frac{x+1}{\log(x+1)^3} = \frac{1}{0_-} = -\infty$

$\lim_{x \to 0^+} \frac{x+1}{\log(x+1)^3} = \frac{1}{0_+} = +\infty$

asymptote at $+\infty$?

$$m = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{(x+1)^{\frac{1}{3}}}{x \log^3(x+1)} =$$

$y = \lg(x+1)$

$x = e^y - 1$

$$= \lim_{y\to+\infty} \frac{e^{y/3}}{(e^y-1)\,y} =$$

$$= \lim_{y\to+\infty} \frac{e^y}{e^y-1}\,\frac{e^{y/3}}{e^y\,y} = \lim_{y\to+\infty} 1\cdot\frac{1}{e^{\frac{2}{3}y}\,y} = 0$$

$$q = \lim_{x\to+\infty} f(x) - 0x = \lim_{x\to+\infty} f(x) = +\infty$$

$\Longrightarrow$ there is not an asymptote at $+\infty$.

The function is differentiable at each point $x \in]-1, +\infty[\setminus \{0\}$ because it is composition of differentiable functions. Hence, in particular, it is continuous on its domain.

Let us compute the derivative: $(x) =$

$$= \left(\frac{(x+1)^{\frac{1}{3}}}{\log^3(x+1)} \right)' =$$

$$= \frac{\frac{1}{3}(x+1)^{-\frac{2}{3}} \log^3(x+1) - 3\log^2(x+1)\frac{(x+1)^{\frac{1}{3}}}{x+1}}{\log^6(x+1)} =$$

$$= \frac{1}{3}(x+1)^{-\frac{2}{3}} \log^{-4}(x+1) \left(\log(x+1) - 9\right)$$

Hence

$f'(x) \geq 0 \iff \log(x+1) \geq 9$

$\iff x+1 \geq e^9 \iff x \geq e^9 - 1$

$\Longrightarrow \bar{x} = e^9 - 1$ is a relative minimum point.

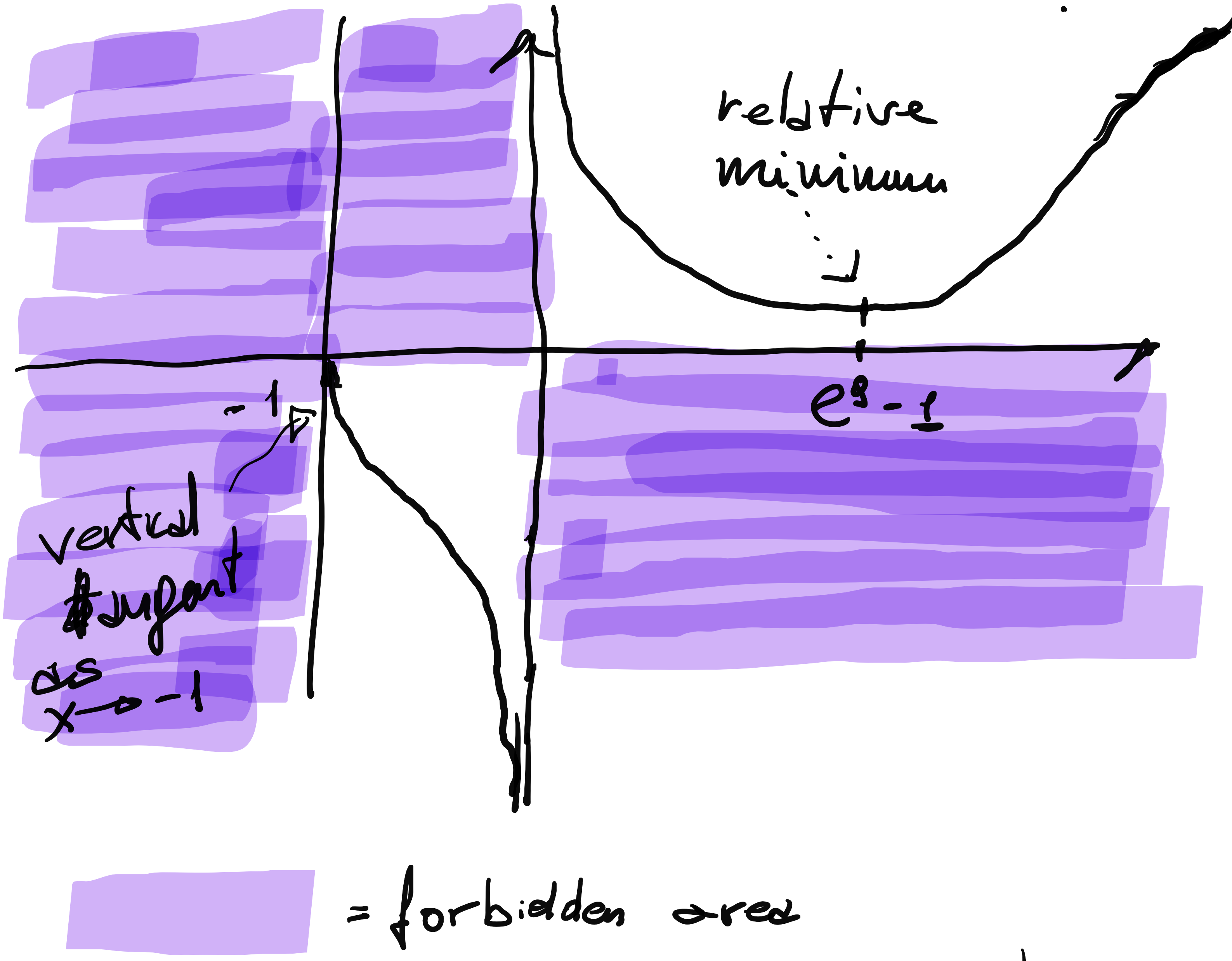
$$f(e^9-1) = \frac{(e^9)^{\frac{1}{3}}}{9^3} = \frac{e^3}{729}$$

$$\lim_{x\to -1} f'(x) = \lim_{x\to -1} \frac{\log(x+1) - 9}{3(x+1)^{\frac{2}{3}} \log^4(x+1)} =$$

$$\underset{y=\log(x+1)}{=} \lim_{y\to -\infty} \frac{y-9}{3e^{\frac{2}{3}y}\, y^4} =$$

$$\underset{z=-y}{=} \lim_{z\to +\infty} - \frac{(z+9)\, e^{\frac{2}{3}z}}{3 z^4} \overset{H}{=}$$

$$\lim_{z\to +\infty} - \frac{e^{\frac{2}{3}z} + (z+9)\, e^{\frac{2}{3}z}}{12 z^3} \overset{H}{=} \dots \overset{H}{=} \dots \overset{H}{=} -\infty$$



To compute convexity we should study the sign of f''.

Probably it is convex on an interval $]-1,\xi[$ with $\xi<0$, and concave on $]\xi,0[$.

Similarly it should be convex on an interval

$]0, \eta[$, for some $\eta > 0$,
and concave on $]\eta, +\infty[$