COVER INEQUALITIES

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VALID INEQUALITIES FOR (M)ILP

	GENERAL PURPOSE CUTS	CUSTOMIZED CUTS
CONVEX HULL YES	GOMORY (ILP, MILP) ...	ODD CUTS (WEIGHTED MATCHING) ...
CONVEX HULL NO	...	NON-AGGREGATED (FACILITY LOCATION) COVER (MULTI-KNAPSACK) ...

THE KNAPSACK PROBLEM 0/1

$$\left.\begin{aligned} \max \ & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \ & \sum_{i=1}^{n} a_i x_i \le \beta \\ & 0 \le x_i \le 1 \quad i=1..n \\ & x_i \in \mathbb{Z} \end{aligned}\right\} X$$

p_i : profit of item i

a_i : weight of " "

β : capacity of K.

$x_i = \begin{cases} 1 & \text{item } i \text{ selected} \\ 0 & \text{otherwise} \end{cases}$

$a_i, \beta \in \mathbb{Z}$ (or $\mathbb{Q} \rightsquigarrow$ l.c.d.)

$\boxed{z_I > z_L}$ and can we strengthen by adding valid inequalities?

- YES 1 : (should know)
- YES 2 : COVER INEQUALITIES ...

COVER INEQUALITIES FOR THE KNAPSACK PROBLEM 0/1

COVER : $C \subseteq \{1..n\}$ such that $\sum_{i \in C} a_i > \beta \quad \equiv \quad \sum_{i \in C} a_i \ge \beta + 1$

! CANNOT SELECT ALL THE ITEMS OF C !

$$x \in X \implies \sum_{i \in C} x_i < |C| \quad \equiv \quad \boxed{\sum_{i \in C} x_i \le |C| - 1}$$

COVER INEQUALITY, VALID FOR X

EQUIVALENT FORM :

$$|C| \equiv \sum_{i \in C} 1 \to \sum_{i \in C} (x_i - 1) \le -1 \implies \boxed{\sum_{i \in C} (1 - x_i) \ge 1}$$

▶ BETTER FORMULATION : ADD $\sum_{i \in C} (1 - x_i) \ge 1, \ \forall \underbrace{C \subseteq \{1..n\}}_{O(2^n)} : \sum_{i \in C} a_i \ge \beta + 1$ (*)

$\implies$ FIND !

SEPARATION PROCEDURE

(SOLVE THE LINEAR RELAXATION AND OBTAIN $\bar{x} \notin \mathbb{Z}^n$)

SEPARATION PROBLEM: GIVEN $\bar{x} \notin \mathbb{Z}^n$, FIND (OR STATE $\nexists$) $C \subseteq \{1 \dots n\}$:

(i) $\sum_{i \in C} a_i \geq \beta + 1$

AND

(ii) $\sum_{i \in C} (1 - \bar{x}_i) < 1$

OPT. PROBLEM

DECISION VARIABLE $z_i = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{otherwise} \end{cases}$

$\hookrightarrow \sum_{i \in C} \square = \sum_{i=1}^{n} \square z_i$

FORMULATION OF THE SEPARATION PROBLEM AS AN ILP:

$$w^* = \min \sum_{i=1}^{n} (1 - \bar{x}_i) z_i$$
$$\text{s.t.} \quad \sum_{i=1}^{n} a_i z_i \geq \beta + 1$$
$$x \in \{0,1\}$$

- IF $w^* < 1$, ADD COVER INEQUALITY FOR $C = \{i = 1..n \mid z_i^* = 1\}$
  OTHERWISE NO VIOLATED COVER INEQUALITY EXISTS

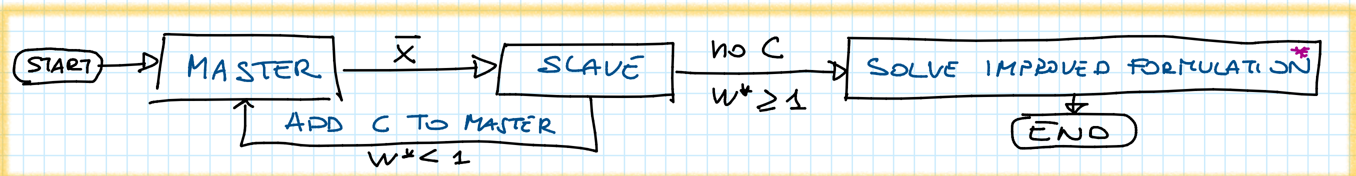
SEPARATION PROCEDURE: INTEGRATION INTO A CUTTING PLANE PROCEDURE

MASTER: SOLVE THE LINEAR RELAXATION

SLAVE: SOLVE THE SEPARATION PROBLEM

TRANSFORM INTO A KP-0/1 : $z_i = 1 - y_i$

- $\min \sum_{i=1}^{n} (1 - \bar{x}_i) z_i \sim \max - \sum_{i=1}^{n} (1 - \bar{x}_i)(1 - y_i) =$
  $= - \sum_{i=1}^{n} (1 - \bar{x}_i) + (1 - \bar{x}_i) y_i \sim \max \sum_{i=1}^{n} (1 - \bar{x}_i) y_i$
- $\sum_{i=1}^{n} a_i z_i = \sum_{i=1}^{n} a_i (1 - y_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i y_i \geq \beta + 1 \Rightarrow$
  $\Rightarrow \sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i - \beta - 1$ (Notice $\sum_{i=1}^{n} a_i > \beta$)



* USE, E.G., B&B OR CUTTING PLANE WITH GOMORY CUTS (WE DO NOT KNOW IF THE IMPROVED FORMULATION IS IDEAL!

COVER INEQUALITIES FOR MORE GENERAL BINARY PROBLEMS

$$\max \sum_{i=1}^{n} c_i x_i = c^T x$$

$$\left.\begin{array}{ll} \text{s.t.} & \sum_{i=1}^{n} a_{ki} x_i \le b_k, \quad k = 1..m \\ & 0 \le x_i \le 1, \quad i = 1..n \end{array}\right\} P$$

$$x_i \in \mathbb{Z}$$

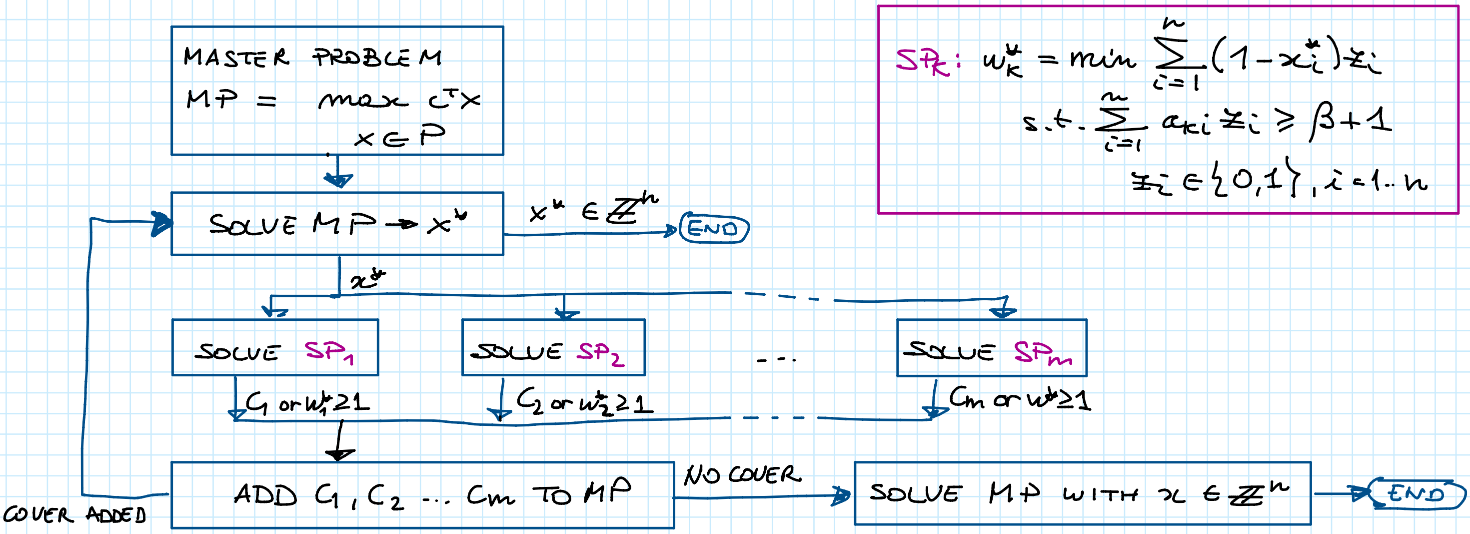
$a_{ki}, b_k \ge 0, \ \forall k, i$

$\Rightarrow$ MULTI KP-0/1

$$SP_k: \ w_k^* = \min \sum_{i=1}^{n} (1 - x_i^*) z_i$$

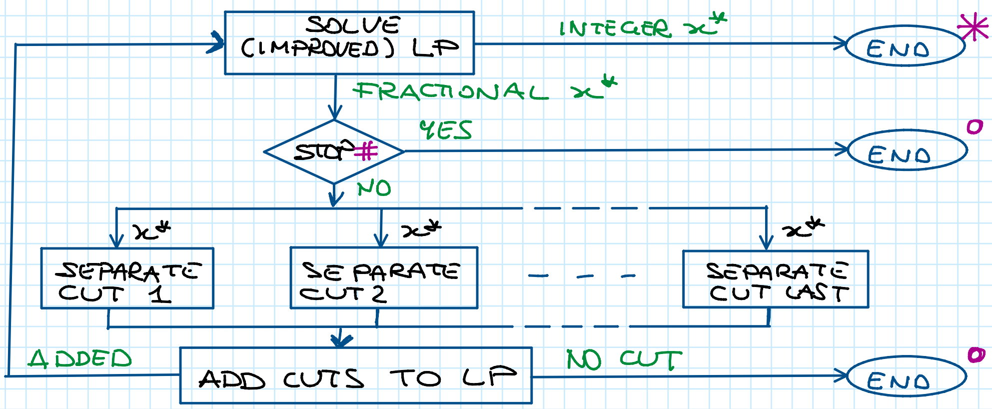
$$\text{s.t.} \sum_{i=1}^{n} a_{ki} z_i \ge \beta + 1$$

$$z_i \in \{0,1\}, \ i = 1..n$$



GENERAL CUTTING PLANE PROCEDURE

$$\left.\begin{array}{l} \left.\begin{array}{l} \max \ c^T x \\ Ax \ge b \end{array}\right\} LP \\ x_i \in \mathbb{Z}, i \in I \end{array}\right\} ILP$$



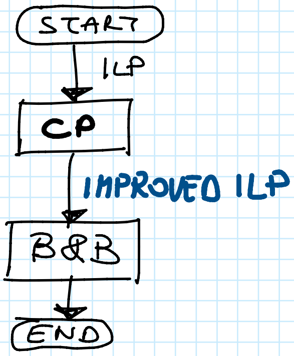
E.G.: MAX TIME, MAX ITER, MINIMUM U.B. IMPROVEMENT, ...

* CONVERGENCE TO OPTIMAL INTEGER SOLUTION [GUARANTEE IF CUT1 ... LAST PROVIDE THE IDEAL FORMULATION (AND # IS OMITTED)]

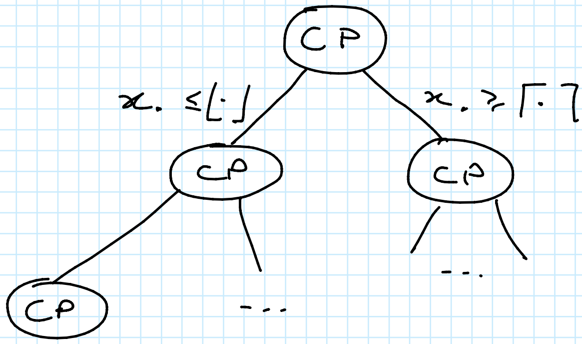
o EXIT WITH IMPROVED FORMULATION

CP = CUTTING PLANE AS ABOVE, RETURNING AN IMPROVED FORMULATION
B&B = BRANCH & BOUND TO FIND AN INTEGER OPTIMAL SOLUTION

CUT-AND-BRANCH



BRANCH-AND-CUT



- CP takes time but it improves bounds! (TRADE-OFF NEEDED)