MATH4U - 24/11/2022

Thursday, 24 November 2022 12:25

1) QUESTIONS ABOUT PREVIOUS EXERCISES

2) STUDY OF A FUNCTION

EXERCISE FROM 27/10/2022 LECTURE

13) $\approx \lim_{n\to 0} -\frac{1}{2}\,\frac{n^2 \to 0}{\sqrt{1+n^2}-1 \to 0}\cdot\frac{\sqrt{1+n^2}+1}{\sqrt{1+n^2}+1} = \lim_{n\to 0} -\frac{1}{2}\,\frac{n^2\left(1+\sqrt{1+n^2}\right)}{\cancel{1}+n^2\cancel{-1}}$

$= \lim_{n\to 0} -\frac{1}{2}\,\frac{\cancel{n^2}\left(1+\sqrt{1+n^2}\right)}{\cancel{n^2}} = -\frac{1}{2}\cdot(1+1) = -1$

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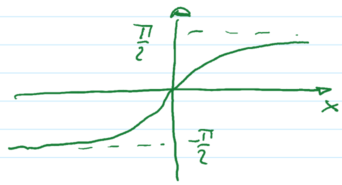
2) $f(x) = x + 2\,\mathrm{arctg}\,\frac{1}{x} + \pi$



$D[f] = x \neq 0 \to D = \mathbb{R}-\{0\}$

$\mathrm{sgn}(f):\ x + 2\,\mathrm{arctg}\,\frac{1}{x} + \pi > 0$

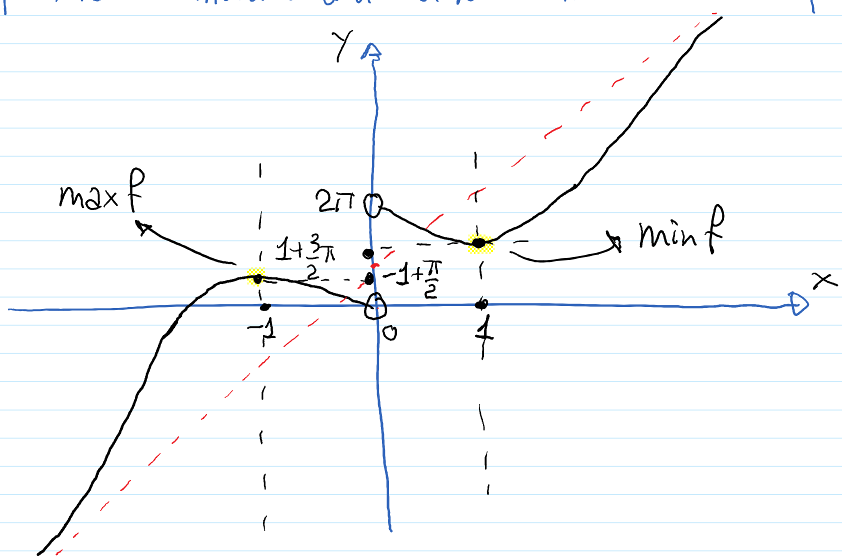
$\lim_{x\to+\infty} x + 2\,\mathrm{arctg}\,\frac{1}{x} + \pi = +\infty + 2\cdot 0 + \pi = +\infty$

$\lim_{x\to-\infty} f(x) = -\infty + \dots = -\infty$

$\lim_{x\to 0^+} f(x) = 2\,\mathrm{arctg}(+\infty) + \pi = \pi + \pi = 2\pi$

$\lim_{x\to 0^-} f(x) = 2\,\mathrm{arctg}(-\infty) + \pi = -\pi + \pi = 0$

$\to f(x)$ is continuous and derivable in $D = \mathbb{R}-\{0\}$



$f'(x) = \frac{d}{dx}\left[x + 2\,\mathrm{arctg}\,\frac{1}{x} + \pi\right] = 1 + 2\,\frac{1}{\ \ }\left(-\frac{1}{\ }\right)$

$$f'(x) = \frac{d}{dx}\left[x + 2\,\mathrm{arctg}\,\frac{1}{x} + \pi\right] = 1 + 2\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)$$

$$= 1 + 2\,\frac{\cancel{x^2}}{x^2+1}\left(-\frac{1}{\cancel{x^2}}\right) = 1 - \frac{2}{1+x^2} = \frac{-1+x^2}{1+x^2}$$

$\mathrm{sgn}(f')$: $D \geq 0$ $\qquad f' \geq 0 \quad x \leq -1 \vee x \geq 1$

$N \geq 0 \Leftrightarrow x^2 - 1 \geq 0 \qquad f' < 0 \quad -1 < x < 1$

$\Longrightarrow$ f increasing in $]-\infty; -1] \cup [1; +\infty[$

f decreasing in $[-1; 0[\cup]0; 1]$

$f(1) = 1 + 2\,\mathrm{arctg}\,1 + \pi = 1 + \frac{\pi}{2} + \pi = 1 + \frac{3}{2}\pi$

$\rightarrow \pi + \left(\frac{\pi}{2} + 1\right) < 2\pi$

~ 2.5

$f(-1) = -1 + 2\,\mathrm{arctg}\,{-1} + \pi = -1 + \frac{\pi}{2} > 0$

ASYMTOTES

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x + 2\,\mathrm{arctg}\,\frac{1}{x} + \pi}{x} = \lim_{x\to+\infty} \frac{x + o(x)}{x} = 1 = \lim_{x\to-\infty} \frac{f(x)}{x}$$

$$\lim_{x\to+\infty} f(x) - 1\cdot x = \lim_{x\to+\infty} 2\,\mathrm{arctg}\,\frac{1}{x} + \pi = \pi = \lim_{x\to-\infty} f(x) - 1\cdot x$$

$\rightarrow$ OBLIQUE ASYMTOTE $y = x + \pi$

$$f''(x) = \frac{d}{dx}\left(1 - \frac{2}{x^2+1}\right) = \frac{d}{dx}\left(-2\,(x^2+1)^{-1}\right) = +2\,(x^2+1)^{-2}\cdot 2x = \frac{4x}{(x^2+1)^2}$$

$$\frac{d}{dx} f(x)^{\alpha} = \alpha\, f(x)^{\alpha-1} f'(x)$$

$f'' \geq 0 \Leftrightarrow x \geq 0$ $\qquad$ f convex in $]0; +\infty[$

f concave in $]-\infty; 0[$

16) $f(x) = e^{-x}\,|e^x - x^2|$

$$= e^{-x}(e^x - x^2)\,\mathrm{sgn}(e^x - x^2)$$

$$= (1 - x^2 e^{-x})\,\mathrm{sgn}(e^x - x^2)$$

$$\frac{d}{dx}|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} = \mathrm{sgn}(x)$$

$|x| = x \cdot \mathrm{sgn}(x)$

$D = \mathbb{R}$

$\mathbb{D} = \mathbb{R}$

$\operatorname{sgn}(f(x)) = +1$, $\underbrace{e^x = x^2}$

$\operatorname{sgn}(e^x - x^2) = \begin{cases} +1 & x > \alpha \\ -1 & x < \alpha \end{cases}$

$\lim_{x \to +\infty} (1 - x^2 e^{-x})(+1) = 1 \longrightarrow$ HORIZONTAL ASYMTOTE

$\lim_{x \to -\infty} (1 - x^2 e^{-x})(-1) = (-\infty)(-1) = +\infty$

$f'(x) = \frac{d}{dx}\Big((1 - x^2 e^{-x}) \operatorname{sgn}(e^x - x^2)\Big) = \operatorname{sgn}(e^x - x^2)\left[-2x e^{-x} + (-x^2)(-e^{-x})\right]$

$\frac{d}{dx} \operatorname{sgn} x = 0 \longrightarrow \frac{d}{dx} f(x)\operatorname{sgn}(x) = f'(x)\operatorname{sgn}(x) + \cancel{f(x)\operatorname{sgn}'(x)}$

$= \operatorname{sgn}(e^x - x^2)\left[e^{-x}(x^2 - 2x)\right]$

$f'(x) > 0$

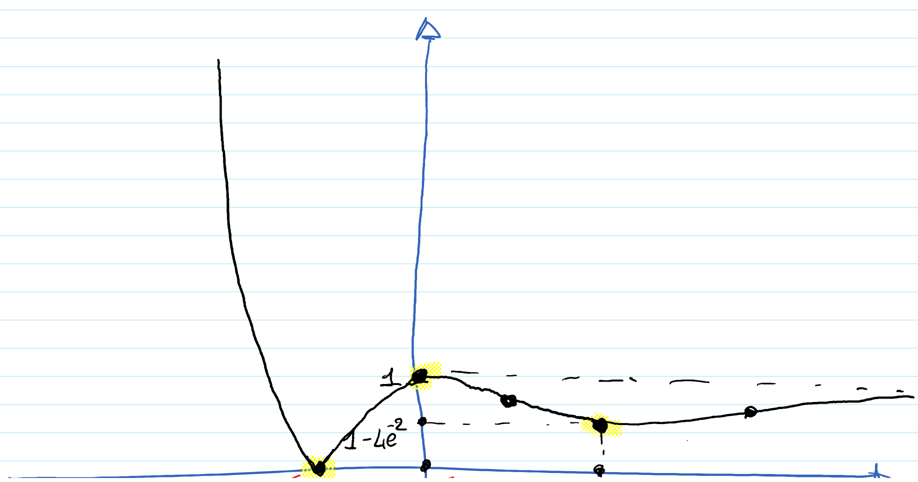
		α		0		2	
e^{-x}	+		+		+		+
$x^2 - 2x$	+		+	0	−	0	+
sgn()	−	∄	+		+		+
f'	−	∄	+	0	−	0	+

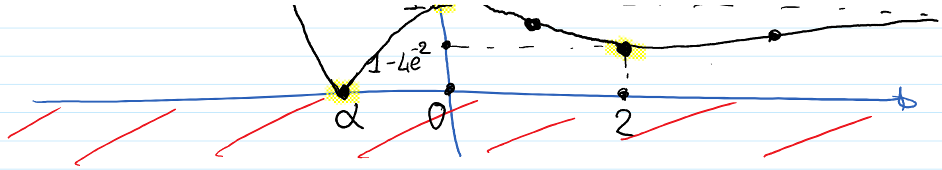
f continuous in $\mathbb{R}$

f derivable in $\mathbb{R} \setminus \{\alpha\}$

$\lim_{x \to \alpha^+} \underbrace{e^{-x}}_{>0} \underbrace{(x^2 - 2x)}_{>0} = \beta > 0$

$\lim_{x \to \alpha^-} f'(x) = -\beta < 0$





$f(\alpha) = \underset{=0}{e^{-\alpha}|e^{\alpha}-\alpha^2|}$ with α solution of $e^x - x^2$ $\longrightarrow$ min f (global)

$f(0) = e^{-0}|e^0 - 0^2| = 1|1-0| = 1$ $\longrightarrow$ max f

$f(2) = 1 - 4e^{-2} < 1$ $\longrightarrow$ min f

$\Big\}$ local

12) $f(x) = \log \dfrac{|x^2-3|}{x+1}$

$\mathbb{D}$: $\dfrac{|x^2-3|}{x+1} > 0 \iff \begin{cases} x > -1 \\ x^2 - 3 \neq 0 \iff x \neq \pm\sqrt{3} \end{cases}$

$\mathbb{D}[f] = \cancel{]-1;+\infty[\setminus \{\pm\sqrt{3}\}}$

$]-1;+\infty[\setminus \{\sqrt{3}\}$

sgn f: $f(x) > 0 \iff \dfrac{|x^2-3|}{x+1} > 1$ in $\mathbb{D}$ $x+1 > 0$

$\Downarrow$

$|x^2-3| > x+1$

$\left[\begin{array}{l} \begin{cases} x > \sqrt{3} \\ x^2 - 3 > x + 1 \end{cases} \\ \vee \\ \begin{cases} -1 < x < \sqrt{3} \\ -x^2 + 3 > x + 1 \end{cases} \end{array}\right.$

$x^2 - x - 4 > 0$ $\quad x_{12} = \dfrac{1 \pm \sqrt{1+16}}{2} = \dfrac{1 \pm \sqrt{17}}{2}$

$-x^2 - x + 2 > 0$

$x^2 + x - 2 < 0$

$x_{12} = \dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm 3}{2} = -2; 1$

$\begin{cases} x > \sqrt{3} \\ x < \underbrace{\dfrac{1-\sqrt{17}}{2}}_{<0} \vee x > \dfrac{1+\sqrt{17}}{2} \end{cases}$

$\Downarrow$

$\dfrac{1}{2} + \sqrt{\dfrac{17}{4}} \sim \dfrac{1}{2} + \sqrt{4} \sim \dfrac{1}{2} + 2 > \sqrt{3}$

$\begin{cases} -1 < x < \sqrt{3} \\ -2 < x < 1 \end{cases}$ $\quad \boxed{-1 < x < 1}$

$\boxed{x > \dfrac{1+\sqrt{17}}{2}}$

$\hookrightarrow$ $f(x) > 0$ in

$$]-1,1[\ \cup\ ]\frac{1+\sqrt{17}}{2};+\infty[$$

$$\lim_{x\to+\infty}\log\frac{|x^2-3|}{x+1}=\log+\infty=+\infty$$

$\to$ DON'T HAVE ASYMTOTES

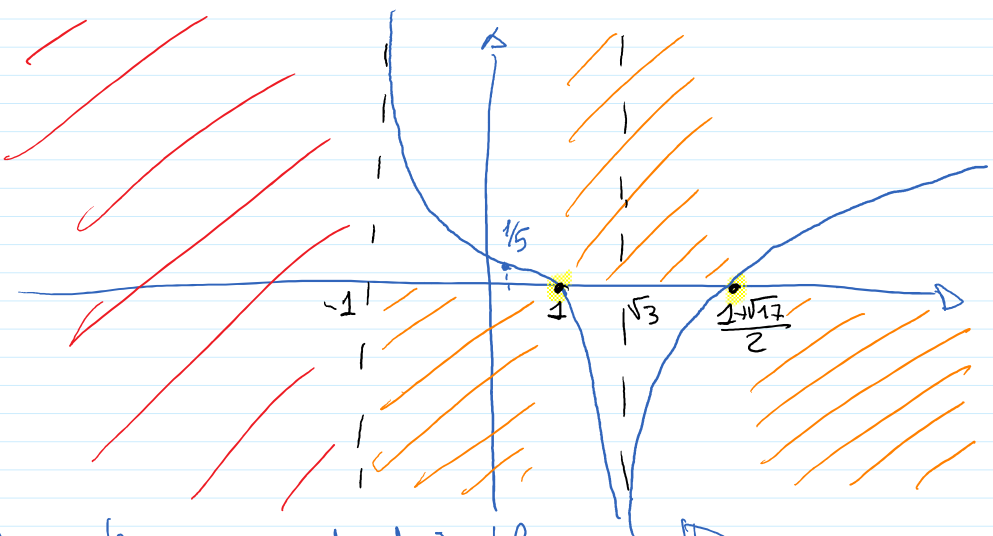
$f(x)\sim\log x \quad \begin{cases}\lim \frac{\log x}{x}=0\\ \lim \log x-0x=+\infty\end{cases}$

$$\cancel{\lim_{x\to-\infty}} \quad \lim_{x\to-1^+}\log\frac{|x^2-3|}{x+1}=\lim+\infty=+\infty$$

$\to$ VERTICAL ASYMTOTE $x=-1$

$$\lim_{x\to\sqrt{3}^\pm}\log\frac{|x^2-3|}{x+1}=\lim 0^+=-\infty$$

$\to$ VERTICAL ASYMTOTE $x=\sqrt{3}$



f continuous and derivable in D

$$f'(x)=\frac{d}{dx}\log\frac{|x^2-3|}{x+1}$$

$$x>\sqrt{3}:\ \frac{d}{dx}\log\frac{x^2-3}{x+1}=\frac{\cancel{x+1}}{x^2-3}\cdot\frac{2x(x+1)-(1)(x^2-3)}{(x+1)^{\cancel{2}}}$$

$$=\frac{2x^2+2x-x^2+3}{(x^2-3)(x+1)}=\frac{x^2+2x+3}{(x^2-3)(x+1)}$$

$f'>0$

$x^2-3>0\ \forall x>\sqrt{3}$

$x+1>0\ \forall x>\sqrt{3}$

$x^2+2x+3>0 \qquad x_{12}=-1\pm\sqrt{1-3}$

$\Delta<0$

$\to f'>0\ \forall x>\sqrt{3}$

$x<\sqrt{3}:\ \frac{d}{dx}\log\frac{3-x^2}{x+1}$ .... $f'<0\ \{x<\sqrt{3}\}\cap D$

$f' = 0$ ... NEVER

(f'')

ADDITIONAL EXERCISES: STUDY THE FUNCTIONS

18) $f(x) = \log\left(\frac{|x^2-5|}{x+1}\right)$

19) $f(x) = \log|2e^{2x}-3|$

20) $f(x) = \frac{x|x|}{|x-1|}$

1) $f(x) = x^2\sqrt[3]{x+1}$

3) $f(x) = e^x\left(\frac{5x-3}{x^2+2x-3}\right)$

4) $f(x) = \log\left((2-x^2)(1+x)\right)$