Theorem (Lagrange)

$f : [a, b] \longrightarrow \mathbb{R}$, continuous

and differ. on $]a, b[$, $\exists \xi \in]a, b[$

s.t. $\dfrac{f(b) - f(a)}{b - a} = f'(\xi)$

Theorem (Cauchy)

$f : [a, b] \longrightarrow \mathbb{R}$ same as in Lagrange

$g : [a, b] \longrightarrow \mathbb{R}$ " f "

$g'(x) \neq 0 \quad \forall x \in]a, b[$, $g(b) \neq g(a)$

Then $\exists \xi \in]a, b[$

s.t. $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(\xi)}{g'(\xi)}$

Hôpital's rules

Problem: limits of the form $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ that are $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Theorem I interval, let $x_0 \in \text{int}(I)$, let f, g differtiable on $I \setminus \{x_0\}$

$f, g : I \setminus \{x_0\} \longrightarrow \mathbb{R}$

let f, g be infinitesimal at x_0, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$

Assume that $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell$

Thesis: $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = l = \left(\lim \frac{f'(x)}{g'(x)}\right)$

<u>Proof</u>: $x_0 \in \mathbb{R}$

$\displaystyle\lim_{x\to x_0^+} \frac{f(x)}{g(x)}$

We can extend continuously f, g to $[x_0\ x]$, by setting $f(x_0)=0$, $g(x_0)=0$

Apply Cauchy th. to

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0} \frac{f'(\xi_x)}{g'(\xi_x)} = \lim_{x\to x} \frac{f'(x)}{g'(x)}$$

$\xi_x \in]x_0, x[$. When $x \to x_0$ $\xi_x = \xi$

l

q.e.d

$$\lim_{x\to 0} \frac{tg(x^2)}{\sin^2 x} \overset{H}{=} \frac{\frac{1}{\cos^2(x^2)}\cdot 2x}{2\sin x \cos x} = \lim \underbrace{\frac{2x}{2\sin(x)}}_{1} \frac{\frac{1}{\cos^2(x^2)}}{\cos(x)} = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{1+x^2-e^{x^2}+(\sin x)^3}$$

Compute

$$\lim_{x\to x_0}\frac{f'}{g'} = \lim_{x\to x_0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{2x - \underbrace{e^{x^2}\cdot 2x} + 3(\sin x)^2\cos x}$$

Compute

$$\lim_{x\to x_0}\frac{(f')'}{(g')'} = \lim_{x\to x_0} \frac{-\sin x - x\cos x}{2-2xe^{x^2}\cdot 2x - 2e^{x^2} + \overbrace{3\cdot 2\sin x\cos x\cos x - 3(\sin x)^2\sin x}}$$

$$\lim_{x\to 0} \frac{x\left(1-\frac{x^2}{2}+o(x^2)\right) - \left(x-\frac{x^3}{6}+o(x^3)\right)}{1+x^2-\left(1+x^2+\frac{x^4}{2}+o(x^4)\right) + \underbrace{(x+o(x))^3}}$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{2} + o(x^3) - \cancel{x} + \frac{x^3}{6} + o(x^3)}{\underbrace{\underbrace{\frac{x^4}{2}}_{o(x^3)} + o(x^4)}_{o(x^3)} + x^3 + \underbrace{o(x^3)}_{o(x^3)}}$$

$$e^y = 1+y+\frac{y^2}{2}+o(y^2)$$

$$= \lim_{x \to 0} \frac{-\frac{1x^3}{3} + o(x^3)}{x^3 + \underline{o(x^3)}} =$$

$$= -\frac{1}{3} .$$

$$\lim_{x \to 0} \frac{e^{\sin x} - 1 - x}{x^2} \overset{H}{=}$$

$$= \lim_{x \to 0} \frac{\cos x \, e^{\sin x} - 1}{2x} \overset{H}{=}$$

$$= \lim_{x \to 0} \frac{-\sin x \, e^{\sin x} + \cos^2 x \, e^{\sin x}}{2} = \frac{1}{2}$$

Definition: $f: D \to \mathbb{R}$

$x_0 \in \text{int}(D)$, f differentiable on a neighborhood of x_0.

$$f''(x_0) := (f')'(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

$f(x) = \sin x \quad f'(x) = \cos x \quad f''(x) = -\sin x$

notation $\quad f''(x_0) = f^{(2)}(x_0) = \frac{d^2 f}{dx^2}$

it is called the second derivative of f.

In general we define the n-th derivative as

$$f^{(n)}(x_0) = \left(f^{(n-1)}\right)'(x_0)$$

$(\sin)^{(2)}(x) = -\sin(x) \qquad (\sin)^{(3)}(x) = -\cos(x)$

$\sin^{(4)}(x) = \sin(x)$

$f(x) = x|x|$

$x \neq 0 \quad f'(x) = \begin{cases} 2x & \text{if } x>0 \\ -2x & \text{if } x<0 \end{cases}$

$f'(0) = \lim_{x\to 0} \frac{\cancel{x}|x|}{\cancel{x}} = 0$

$f''_+(0) = \lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x} = \lim \frac{2x}{x} = 2$

$f''_-(0) = \lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0^-} \frac{-2x}{x} = -2$

no second derivative

PROBLEM

We know that the existence of $f'(x_0)$

$$\underbrace{f(x)} = \underbrace{f(x_0) + f'(x_0)(x-x_0)} + \underbrace{o(x-x_0)}$$

$$f(x) \overset{?}{=} C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + o((x-x_0)^2)$$

$$f(x) \overset{?}{=} \underbrace{c_0 + c_1(x-x_0)} + c_2(x-x_0)^2 + \dots + c_n(x-x_0)^n + o\left((x-x_0)^n\right)$$

$$p(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots + c_n(x-x_0)^n$$

$$p'(x_0) = c_1 + 2c_2(x-x_0)\Big|_{x=x_0} + \dots + n c_n (x-x_0)^{n-1}\Big|_{x=x_0}$$

$$p'(x_0) = c_1$$

$$p''(x_0) = 2c_2 + 6c_3(x-x_0)\Big|_{x=x_0} + \dots + n(n-1)(x-x_0)^{n-2}\Big|_{x=x_0}$$

$$\Rightarrow p''(x_0) = 2c_2 \Leftrightarrow c_2 = \frac{p''(x_0)}{2! = 2}$$

$$p'''(x_0) = 2 \cdot 3\, c_3 \Leftrightarrow c_3 = \frac{p'''(x_0)}{3! = 6}$$

$$p(x) = p(x_0) + p'(x_0)(x-x_0) + \frac{p''(x_0)}{2!}(x-x_0)^2 + \dots + p^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

We would like, for generic f,

$$(*)\quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o\left((x-x_0)^n\right)$$

If f is differentiable n times then $(*)$ holds true.

$$\sin x = x - \frac{x^3}{6} + \boxed{\frac{x^5}{120}} + o(x^5)$$

This was given.

Let us deduce $(*)$

$x_0 = 0$

$$\sin x = f'(0)x + \cancel{\frac{f''(0)}{2}x} + \boxed{\frac{f'''(0)}{6}x^3} + \frac{f^{IV}(0)}{24}x^{(4)} + \frac{f^{(5)}(0)}{120}x^5 + o(x^5)$$

$$f'(0) = (\sin)'(0) = \cos 0 = 1$$

$$f''(0) = (\sin')'(0) = -\sin 0 = 0$$

$$f'''(0) = (\sin)''(0) = -\cos(0) = -1$$

$$f^{IV}(0) = 0$$

$$f^{(V)}(0) = \cos(0) = 1$$
