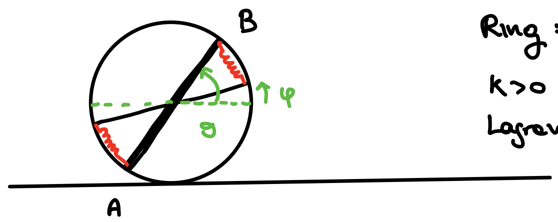
Lesson 25 - 24/11/2022

- Routh method : L_R.
- Conservative central force on the plane.
- Geodesics on the torus.
- 1 EX on Lagrangian formalism.



Bar AB : $2R$, m

Ring : M, R

$k>0$ elastic constant

Lagrangian coo : φ, θ (see Figure).

1) Lagrangian.

2) Equilibria + stability

$L(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{m-1}, \dot{q}_m, t)$

Lagrangian with q_m cyclic coo. As a conseq.

$P_m = \dfrac{\partial L}{\partial \dot{q}_m}$ is a conserved quantity.

$\hookrightarrow \dot{q}_m = u(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{m-1}, t, \underbrace{p_m}_{\text{one parameter.}})$

In particular, we obtain $\dot{q}_m = u(\dots)$ by inverting the $p_m = \dfrac{\partial L}{\partial \dot{q}_m}$.

$L_R(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{m-1}, t) :=$

$:= \left[L(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{m-1}, \dot{q}_m, t) - p_m \dot{q}_m \right] \Big|_{\dot{q}_m = u(\dots)}$

[L and L_R give the same Lagr. eqs for $q_1 - q_{m-1}$.]

<u>Proof</u>

Consider q_h, $h = 1 — n-1$. $\quad \left(\frac{d}{dt}\frac{\partial L_R}{\partial \dot{q}_h} - \frac{\partial L_R}{\partial q_h} = 0 \right)$

$$\frac{\partial L_R}{\partial q_h} = \frac{\partial L}{\partial q_h} + \underbrace{\frac{\partial L}{\partial \dot{q}_m}}_{\overset{\parallel}{p_m}} \underbrace{\frac{\partial \dot{q}_m}{\partial q_h}}_{=} - p_m \underbrace{\frac{\partial \dot{q}_m}{\partial q_h}}_{=} = \frac{\partial L}{\partial q_h}$$

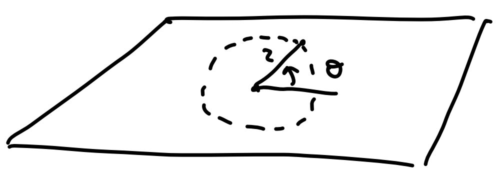
$$\frac{\partial L_R}{\partial \dot{q}_h} = \frac{\partial L}{\partial \dot{q}_h} + \underbrace{\frac{\partial L}{\partial \dot{q}_m}}_{\overset{\parallel}{p_m}} \underbrace{\frac{\partial \dot{q}_m}{\partial \dot{q}_h}}_{=} - p_m \underbrace{\frac{\partial \dot{q}_m}{\partial \dot{q}_h}}_{=} = \frac{\partial L}{\partial \dot{q}_h} \qquad \boxtimes$$

Generalizing to $q_{m+1} — q_n$ cyclic coordinates:

$$L_R(q_1 — q_m, \dot{q}_1 — \dot{q}_m, t) :=$$

$$:= \left[L(q_1 — q_m, \dot{q}_1 — \dot{q}_m, \dot{q}_{m+1} — \dot{q}_n, t) - \sum_{k=m+1}^{n} p_k \dot{q}_k \right] \Bigg|_{(\dot{q}_{m+1} — \dot{q}_n) = w(q_1 — q_m, \dot{q}_1 — \dot{q}_m, t, \underbrace{p_{m+1} — p_n}_{n-m \text{ parameters.}})}$$

EX 1 Conservative central force.



P, m on the plane
Use polar coo. θ, r
Subj. to a conservative central force: $V(r)$.

The Lagrangian: $L(r, \theta, \dot{r}, \dot{\theta}) = K - V(r)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$$

$$\dot{x}^2+\dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) = L(r,\dot{r},\dot{\theta})$$

θ is a cyclic coo $\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = c \Rightarrow \dot{\theta} = c/mr^2$

$\Downarrow$ $\qquad$ c $\parallel$ constant of motion, conservation of angular momentum in the direction of the orthogonal plane.

The reduced Lagrangian is

$$L_R(r,\dot{r}) = L\Big(r,\dot{r},\underbrace{\frac{c}{mr^2}}_{\dot{\theta}=\dots}\Big) - \underbrace{c}_{p_\theta}\underbrace{\frac{c}{mr^2}}_{\dot{\theta}=\dots} =$$

$$= \frac{m}{2}\dot{r}^2 + \underbrace{\frac{\cancel{m}}{2}\cancel{r^2}\frac{c^2}{m^{\cancel{2}}r^{\cancel{4}2}}}\, - V(r) - \underbrace{\frac{c^2}{mr^2}}$$

$$= \underbrace{\frac{m\dot{r}^2}{2}}_{\text{Kinetic energy.}} - \frac{1}{2}\frac{c^2}{mr^2} - V(r) = \frac{m\dot{r}^2}{2} - \left[V_R(r)\right]$$

Lagrange eqs. for L_R are:

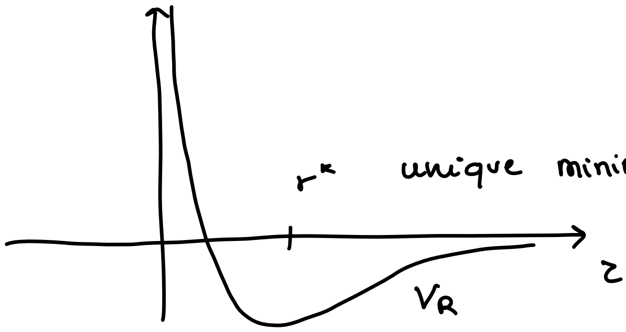
$$\underbrace{\frac{d}{dt}(m\dot{r})}_{m\ddot{r}} - \frac{c^2}{mr^3} + V'(r) = 0 \qquad \left(m\ddot{r} = -V_R'(r)\right)$$

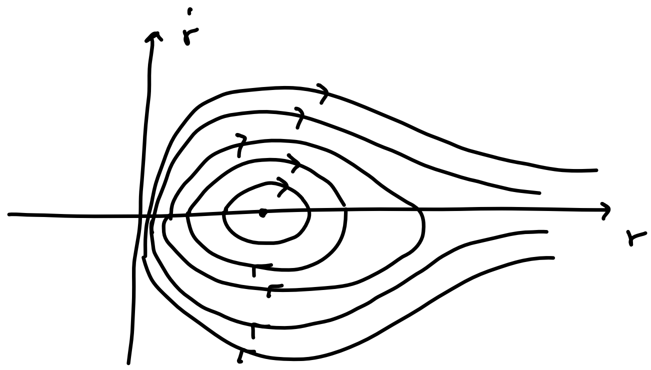
$\swarrow$ a 1-dim. conservative system! We are able to study these systems! By using the conservation of energy and sublevels of $V \leq a$.

$$V(r) = -\frac{k}{r}$$

$$\Rightarrow V_R(r) = -\frac{k}{r} + \frac{1}{2}\frac{c^2}{mr^2}$$

Graph of $V_R(r)$:



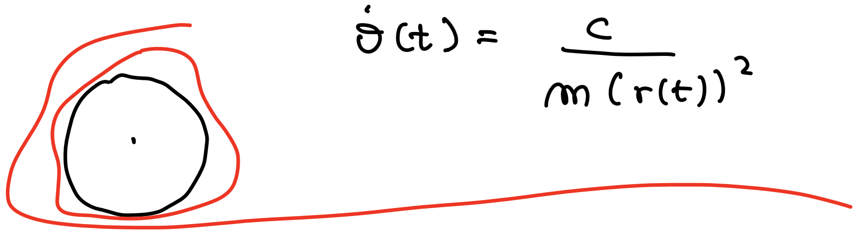


Phase-portrait for the reduced system.

Infos on the original system?

The complete system has a circular uniform motion of radius r^*.

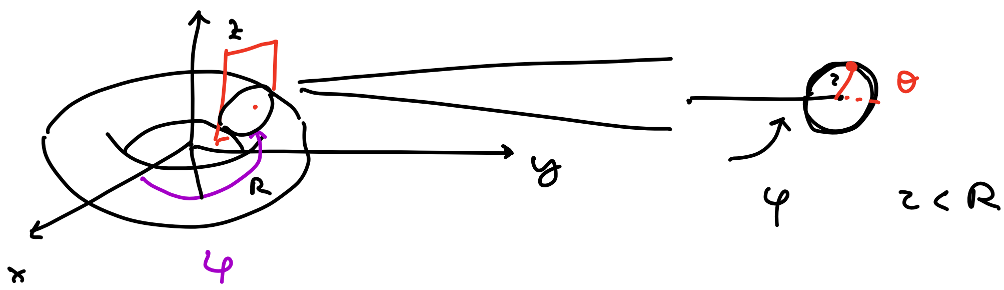
$$\dot\vartheta = \frac{c}{m(r^*)^2} \longrightarrow \begin{cases} \vartheta(t) = \vartheta_0 + \dfrac{c}{m(r^*)^2}\, t \\ r(t) = r^* \end{cases}$$



$$\dot\vartheta(t) = \frac{c}{m(r(t))^2}$$

GEODESICS ON THE TORUS

↳ "Spontaneous motions" → without external forces. $L = K$.



$$(\theta,\varphi) \longmapsto \underset{\overset{\parallel}{\vec{OP}}}{\left(\cos\varphi\,(R+r\cos\theta),\ \sin\varphi\,(R+r\cos\theta),\ r\sin\theta\right)}$$

$$\vec{v}_P = \dot{\vec{OP}} = \dot\varphi\left(-\sin\varphi\,(R+r\cos\theta),\ \cos\varphi\,(R+r\cos\theta),\ 0\right) +$$
$$+\ r\dot\theta\left(-\cos\varphi\sin\theta,\ -\sin\varphi\sin\theta,\ \cos\theta\right).$$

Since $\vec{v}_P$ is the sum of 2 $\perp$ vectors,

$$|\vec{v}_P|^2 = \dot\varphi^2(R+r\cos\theta)^2 + r^2\dot\theta^2$$

$$K = \frac{1}{2}m\left(\dot\varphi^2(R+r\cos\theta)^2 + r^2\dot\theta^2\right) = L(\theta,\dot\theta,\dot\varphi)$$

φ is a cyclic coordinate $\Rightarrow$ we can reduce the study of the dynamics to 1-variable $(\theta,\dot\theta)$

$$\frac{\partial L}{\partial\dot\varphi} = m(R+r\cos\theta)^2\dot\varphi = c$$

$$\Rightarrow \dot\varphi = \frac{c}{m(R+r\cos\theta)^2}$$

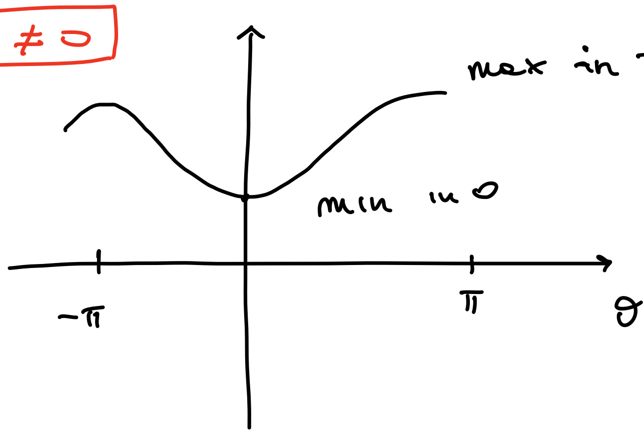
$$L_R(\theta,\dot\theta) = \frac{1}{2}m\left[\frac{c^2}{m^{2}(R+r\cos\theta)^{\cancel{4}\,2}}\cancel{(R+r\cos\theta)^2} + r^2\dot\theta^2\right] -$$

$$- c \cdot \frac{c}{m(R+2\cos\theta)^2} =$$

$$= \underbrace{\frac{1}{2} m r^2 \dot\theta^2}_{\frac{1}{2}(mr^2)\dot\theta^2 \text{ Kinetic energy}} - \underbrace{\frac{c^2}{2m(R+2\cos\theta)^2}}_{V_R(\theta) = \frac{c^2}{2m(R+2\cos\theta)^2}}$$

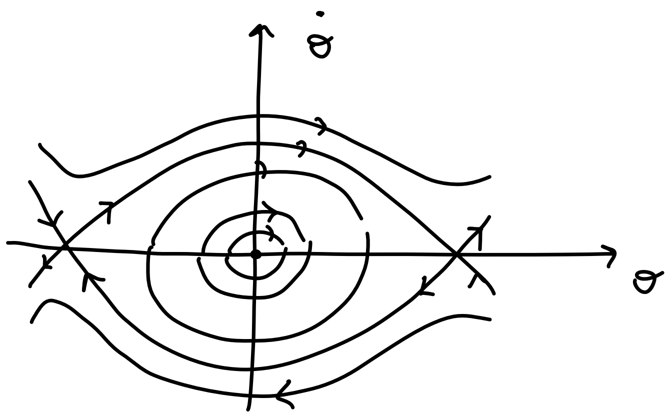
Phase portrait of the reduced 1-dim. system.

$\boxed{c \neq 0}$



$$V_R(\theta) = \frac{c^2}{2m(R+2\cos\theta)^2} \qquad V_R'(\theta) = \frac{c^2 2\sin\theta}{m(R+2\cos\theta)^3} = 0$$

$\theta = 0, \pi.$



Re-construction of the dynamics in the original system.

$\vartheta = 0$: external parallel

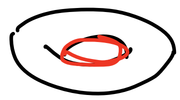
$\vartheta = \pi$: internal parallel

$$\dot\varphi = \frac{c}{m(R + r\cos\vartheta)^2}$$

$$\begin{cases} \vartheta_t \equiv 0 \\ \varphi_t = \varphi_0 + \dfrac{c}{m(R+r)^2}\, t \end{cases}$$

$$\dot\varphi_t = \frac{c}{m(R+r)^2}$$

$$\begin{cases} \vartheta_t \equiv \pi \\ \varphi_t = \varphi_0 + \dfrac{c}{m(R-r)^2}\, t \end{cases}$$



$\boxed{c=0}$

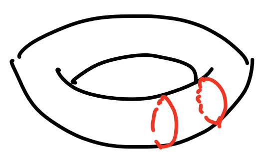
$c=0$ means $\dot\varphi = 0 \Rightarrow \varphi = \text{const} \Rightarrow$ motions on the meridians.

Moreover: $V_R(\vartheta) = 0$

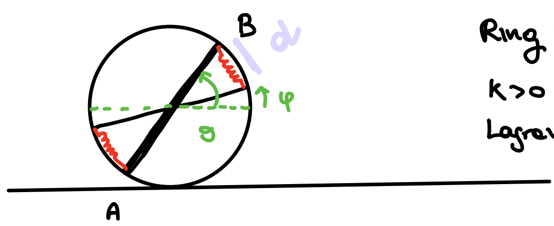
$$e = \frac{1}{2} m r^2 \dot\vartheta^2 + \underbrace{V_R(\vartheta)}_{=0}$$

$$\Rightarrow \frac{2e}{mr^2} = \dot\vartheta^2 \Rightarrow \dot\vartheta = \pm\sqrt{\frac{2e}{mr^2}}$$

$$\begin{cases} \varphi_t \equiv \varphi_0 \\ \vartheta_t = \vartheta_0 \pm \sqrt{\dfrac{2e}{mr^2}}\, t \end{cases}$$



$\boxed{\text{EX}}$



Bar AB : $2R$, m

Ring : M, R

$k>0$ elastic constant

Lagrangian coo : φ, θ (see Figure).

1) Lagrangian.

2) Equilibria + stability

$$K_{RING} = \frac{1}{2} M v^2 + \frac{1}{2} M R^2 \dot{\varphi}^2 \underset{x = R\varphi}{=} \frac{1}{2} M R^2 \dot{\varphi}^2 + \frac{1}{2} M R^2 \dot{\varphi}^2 =$$

$$= M R^2 \dot{\varphi}^2$$

$$= \frac{1}{2} (2MR^2) \dot{\varphi}^2$$

$$K_{BAR} =$$

$$= \frac{1}{2} m \underbrace{v^2}_{(R\dot{\varphi})^2} + \frac{1}{2} m \frac{(2R)^2}{12} \dot{\theta}^2 =$$

$$= \frac{1}{2} m R^2 \dot{\varphi}^2 + \frac{1}{2} m \frac{1}{3} R^2 \dot{\theta}^2$$

$$K_{TOT} = K_{RING} + K_{BAR} =$$

$$= \frac{1}{2} \left[(2M + m) R^2 \dot{\varphi}^2 + \frac{mR^2}{3} \dot{\theta}^2 \right]$$

$$V_{el} = 2 \cdot \frac{1}{2} k d^2 = k \left[R^2 + R^2 - 2R^2 \cos(\theta - \varphi) \right]$$

Carnot theorem

$$= -2kR^2 \cos(\theta - \varphi) + \text{const.}$$

$$L = K_{TOT} - V_{el}$$

Lagrange eqs :

$$\begin{cases} (2M+m)\,\ddot{\varphi} - 2k\sin(\vartheta-\varphi) = 0 \\ \dfrac{m}{3}\,\ddot{\vartheta} + 2k\sin(\vartheta-\varphi) = 0 \end{cases}$$

$E = K_{TOT} + V_{el}$ is conserved!

Equilibria:

$\sin(\vartheta-\varphi)=0 \iff \vartheta=\varphi$ OR $\vartheta=\varphi+\pi$

Stability !?

$V_\varphi = -2kR^2\sin(\vartheta-\varphi)$, $V_\vartheta = 2kR^2\sin(\vartheta-\varphi)$

$V_{\varphi\varphi} = 2kR^2\cos(\vartheta-\varphi)$

$V_{\vartheta\vartheta} = 2kR^2\cos(\vartheta-\varphi)$

$V_{\vartheta\varphi} = V_{\varphi\vartheta} = -2kR^2\cos(\vartheta-\varphi)$

$$V''(\vartheta=\varphi) = 2kR^2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \to \text{Eigenvalues are } \underline{\underline{0}}, 2.$$

$$V''(\vartheta=\varphi+\pi) = 2kR^2\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \text{Eigenvalues } 0, -2$$

unstability!!

TO CHECK STABILITY/UNSTABILITY OF $\vartheta=\varphi$ on Monday...

—x—x—