Exercise 5.8.8

2) $f(x) = \frac{\sin x}{x}$ $D = \mathbb{R} \setminus \{0\}$ $S = D$

$\lim_{x \to 0} \frac{\sin x}{x} = 1 \Rightarrow$ f can be extended to $x_0 = 0$ in a continuous way

3) $f(x) = \frac{\tan(2x)}{\tan(x)}$

Domain of tan(2x)

$D = \{x \in \mathbb{R} : x \neq \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}\} \cap$

$\cap \{x \neq k\pi \ \ k \in \mathbb{Z}\} =$

set where tan x = 0

$= \mathbb{R} \setminus \{\frac{\pi}{4} + \frac{k\pi}{2}, h\pi, \ \ k, h \in \mathbb{Z}\}$

$S = D$

$\lim_{x \to 0} \frac{\tan(2x)}{\tan(x)} = \lim_{x \to 0} \frac{2x + o(x)}{x + o(x)} = 2$

$\lim_{x \to \frac{\pi}{4}^{\pm}} \frac{\tan(2x)}{\tan x} = \mp \infty$

$$\lim_{x \to \frac{3}{4}\pi \pm} \frac{\tan(2x)}{\tan(x)} = \pm \infty$$

$$\lim_{x \to \pi} \frac{\tan(2x)}{\tan(x)} = \lim_{x \to \pi} \frac{\tan(2x - 2\pi)}{\tan(x - \pi)}$$

$$\underset{y = x - \pi}{=} \lim_{y \to 0} \frac{\tan(2y)}{\tan(y)} = 2$$

f can be extended in a continuous way to all points $x_0 \in \{k\pi, k \in \mathbb{Z}\}$

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4. $f(x) = \dfrac{\cos\left(\frac{\pi}{2} x\right)}{x^2 - 1}$

$D = \mathbb{R} \setminus \{\pm 1\}$

$S = D$

$$\lim_{x \to 1} \frac{\cos\left(\frac{\pi}{2} x\right)}{x^2 - 1} \underset{y = x - 1}{=} \lim_{y \to 0} \frac{\cos\left(\frac{\pi}{2}(1+y)\right)}{y(y+1)} =$$

$$= \lim_{y\to 0} \frac{-\sin\left(\frac{\pi}{2}y\right)}{y(y+1)} = \lim_{y\to 0} \frac{-\frac{\pi}{2}y - o\left(\frac{\pi}{2}y\right)}{y(y+1)}$$

$$= \lim_{y\to 0}\left(-\frac{\pi}{2} + \frac{o(y)}{y(y+1)}\right) = -\frac{\pi}{2}$$

$$\lim_{x\to -1} = -\frac{\pi}{2}$$ (because the function is even)

f can be extended continuously to both $x=1$ and $x=-1$

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8) $f(x) = x\log|x|$

$D = \mathbb{R}\setminus\{0\}$ $\quad S = D$

$$\lim_{x\to 0^+} x\log|x| = \lim_{x\to 0^+} x\log(x) \underset{x=e^y}{=} \lim_{y\to -\infty} e^y y \underset{z=-y}{=} \lim_{z\to +\infty} \frac{-z}{e^z} = 0$$

$$\lim_{x \to 0-} f(x) \underset{\text{because } f \text{ is odd}}{=} \lim_{x \to 0-} (-f(-x)) \underset{y=-x}{=}$$

$$\lim_{y \to 0+} (-f(y)) = 0$$

Hence $\lim_{x \to 0} f(x) = 0$

and f can be extended to a continuous function

9) $$f(x) = \frac{\sqrt{x^4 - 3x^2 + 2}}{x}$$

$$D = \{x \in \mathbb{R} : \quad x \neq 0 \quad x^4 - 3x^2 + 2 \geq 0\}$$

$$= \{x \in \mathbb{R} : \quad x^2 = y \quad y^2 - 3y + 2 \geq 0, \; x \neq 0\} =$$

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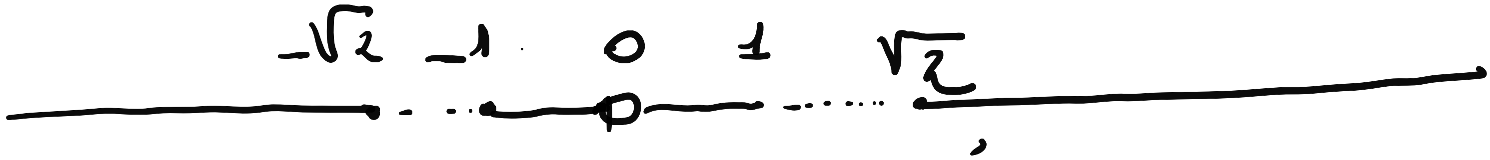
$$= \left\{ x \in \mathbb{R} : \begin{matrix} x^2 = y \\ x \neq 0 \end{matrix} \quad y \geq 2 \right\} \cup$$

$$\left\{ \begin{matrix} x \in \mathbb{R} \\ x \neq 0 \end{matrix} : \quad x^2 = y \quad y \leq 1 \right\}$$

$$]-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty[\cup [-1, 1] \setminus \{0\}$$



$$S = D$$

$$\lim_{x \to 0^{\pm}} f(x) = \pm\infty$$

f <u>cannot</u> be extended to a continuous function at $x = 0$

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14) $f(x) = \sin\left(\dfrac{e^x}{e^{2x} - e^x + 1} \right)$

$$D = \left\{ x \in \mathbb{R} : \quad e^{2x} - e^x + 1 \neq 0 \right\}$$

$= \mathbb{R}$ (because $y = e^x$
$y^2 - y + 1 = 0$
has no solution)

$S = \mathbb{R}$

12) $f(x) = x e^{\frac{x}{x-1}}$

$D = \{ x \in \mathbb{R} , x \neq 1 \} =$
$= \mathbb{R} \setminus \{1\}$

$S = D$

$$\lim_{x \to 1^\pm} x e^{\frac{x}{x-1}} = 1 \cdot e^{\left(\lim \frac{x}{x-1}\right)} =$$

$$= \pm \infty$$

the fonction f cannot be extended continuously at $x = 1$