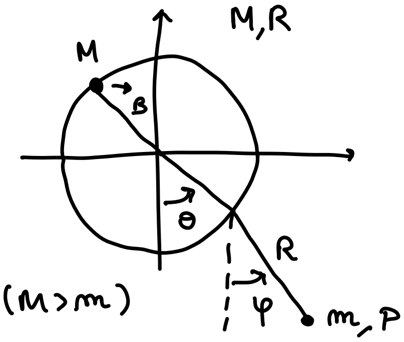
- Ex. of last lesson (Kinetic energy).

$\vec{OP} = (R\sin\theta + R\sin\varphi,\ -R\cos\theta - R\cos\varphi)$

$\Rightarrow \vec{v}_p = R(\dot\theta\cos\theta + \dot\varphi\cos\varphi,\ \dot\theta\sin\theta + \dot\varphi\sin\varphi)$

$\Rightarrow |\vec{v}_p|^2 = R^2(\dot\theta^2 + \dot\varphi^2 + 2\cos(\theta-\varphi)\dot\theta\dot\varphi)$

Position of the point of mass m.

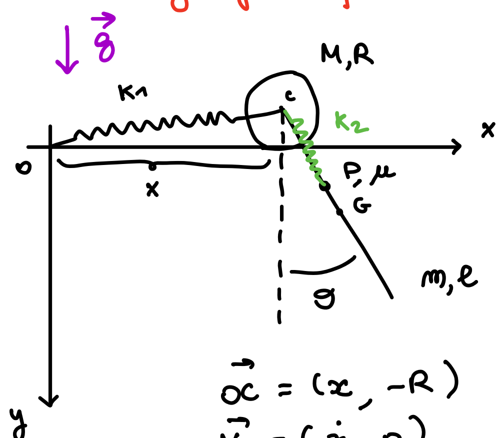


(M>m)

($\exists!$ stable equilibrium : $(\theta^*, \varphi^*) = (\pi, 0)$: from last lecture)

- Partial inversion of L-D Theorem : NON-DEGENERATE HESSIAN THEOREM (without proof).
- 2Ex. on Lagrangian formalism.

$\boxed{1}$

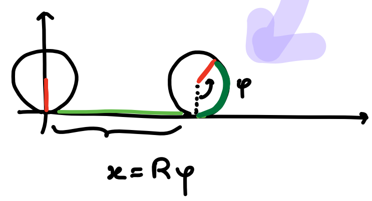


$\vec{OC} = (x, -R)$

$\vec{v}_C = (\dot x, 0)$

$\frac{\text{disc}}{\text{bar}}$: rotates without sliding

$(x + R\varphi = 0)$



- Lagrangian
- Equilibria, stability
- $K_1 = 0$: First integrals.

X = distance from x and x_C.

θ = angle as in the picture.

$s > 0$

$\boxed{2}$ ....

- Routh method, reduced Lagrangian...
(we will conclude tomorrow)

From previous computation:

$K_P = \frac{1}{2} m R^2 (\dot\theta^2 + \dot\varphi^2 + 2\cos(\theta-\varphi)\dot\theta\dot\varphi)$

$K_{RING} = \frac{1}{2} M R^2 \dot\theta^2$

$\downarrow$

Baricenter is fixed

$K_B = \frac{1}{2} M R^2 \dot\theta^2$

$$K_{TOT} = \frac{1}{2} R^2 \left[(2M+m)\dot\vartheta^2 + m\dot\varphi^2 + 2m\cos(\vartheta-\varphi)\dot\vartheta\dot\varphi \right]$$

$$\Downarrow$$

$$a(\vartheta,\varphi) = R^2 \begin{pmatrix} 2M+m & m\cos(\vartheta-\varphi) \\ m\cos(\vartheta-\varphi) & m \end{pmatrix}$$

$$\left[\frac{1}{2} (\dot\vartheta, \dot\varphi)\, \underline{a(\vartheta,\varphi)} \begin{pmatrix} \dot\vartheta \\ \dot\varphi \end{pmatrix} = K_{TOT} \right]$$

Remark: L-D theorem gives (in a general setting) a SUFFICIENT condition for stability of an equilibrium.

$(\Rightarrow)$

$(\not\Leftarrow)$ The problem of necessity is not solved.

A partial inversion of L-D theorem is given by the so-called NON-DEGENERATE HESSIAN THEOREM (no proof).

This theorem is very useful in exercises.

- Constrained system with ideal, fixed and not depending on velocities constraints.
- $Q_h(q) = -\frac{\partial V}{\partial q_h}(q)$ (only conservative forces).
- q^* s.t. $\frac{\partial V}{\partial q_h}(q^*) = 0$ (equilibrium configuration)
- Suppose $\det \frac{\partial^2 V}{\partial q_h \partial q_k}(q^*) \neq 0$.
- THEN $(q^*, 0)$ is stable IFF the quadratic form

$$\frac{\partial^2 V}{\partial q_h \partial q_k}(q^*) \lambda_h \lambda_k > 0$$

(is positive definite)

$\forall \lambda \neq (0 \ldots 0)$.

$\boxed{1}$ $K_{TOT} = K_{DISC} + K_{BAR} + K_{P}$

$\underline{x = R\varphi} \Rightarrow \dot{\varphi} = \dfrac{\dot{x}}{R}$ (disc "purely" rotates)

For the bar : $\dot{\theta}$

$$K_{Disc} = \frac{1}{2} M |\vec{v}_c|^2 + \frac{1}{2} I_c \underbrace{\frac{\dot{x}^2}{R^2}}_{\dot{\varphi}^2} =$$

Konig theorem

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2}\left(\frac{M\cancel{R^2}}{2}\right)\frac{\dot{x}^2}{\cancel{R^2}} = \frac{3}{4} M \dot{x}^2$$

$$K_{BAR} = \frac{1}{2} m \underbrace{|\vec{v}_G|^2} + \frac{1}{2} \underbrace{I_G}_{\frac{ml^2}{12}} \dot{\theta}^2$$

$$\vec{OG} = \left(x + \frac{\ell}{2}\sin\theta,\ \frac{\ell}{2}\cos\theta - R\right)$$

$$\vec{v}_G = \left(\dot{x} + \frac{\ell}{2}\dot{\theta}\cos\theta,\ -\frac{\ell}{2}\dot{\theta}\sin\theta\right)$$

$$|\vec{v}_G|^2 = \dot{x}^2 + \frac{\ell^2}{4}\dot{\theta}^2 + \ell\cos\theta\,\dot{x}\dot{\theta}$$

$$K_{BAR} = \frac{1}{2} m \left[\dot{x}^2 + \frac{\ell^2}{4}\dot{\theta}^2 + \underline{\ell\cos\theta\,\dot{x}\dot{\theta}}\right] + \frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2$$

$$\ell^2\left(\frac{1}{4} + \frac{1}{12}\right) = \frac{\cancel{4}}{\underset{3}{\cancel{12}}}\ell^2$$

$$K_P = ? = \frac{1}{2}\mu|\vec{v}_P|^2$$

$$\vec{OP} = (x + s\sin\theta,\ s\cos\theta - R)$$

$$\vec{v}_P = (\dot{x} + \dot{s}\sin\theta + s\dot{\theta}\cos\theta,\ \dot{s}\cos\theta - s\dot{\theta}\sin\theta)$$

$$|\vec{v}_P|^2 = \dot{x}^2 + \dot{s}^2 + s^2\dot{\theta}^2 + 2\sin\theta\,\dot{x}\dot{s} + 2s\cos\theta\,\dot{x}\dot{\theta}$$

$$K_{TOT} = \underbrace{\frac{3}{4} M \dot{x}^2}_{\text{DISC}} + \underbrace{\frac{1}{2} m \left[\dot{x}^2 + \frac{\ell^2}{3} \dot{\theta}^2 + \ell \cos\theta \, \dot{x}\dot{\theta} \right]}_{\text{BAR}}$$

$$+ \frac{1}{2}\mu \left(\dot{x}^2 + \dot{s}^2 + s^2 \dot{\theta}^2 + 2 \sin\theta \, \dot{x}\dot{s} + 2 s \cos\theta \, \dot{x}\dot{\theta} \right)$$

Potential energy.

$$V = \frac{1}{2} k_1 |\vec{OC}|^2 + \frac{1}{2} k_2 |\vec{CP}|^2 - m g y_G - \mu g y_P$$

$$= \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 s^2 - mg \frac{\ell}{2} \cos\theta - \mu g s \cos\theta + \text{const}$$

$$\boxed{L = K_{TOT} - V}$$

— x —

$$\begin{cases} V_x = k_1 x = 0 \longrightarrow x = 0 \\ V_\theta = \sin\theta \left(mg \frac{\ell}{2} + \mu g s \right) = 0 \\ V_s = -\mu g \cos\theta + k_2 s = 0 \end{cases}$$

$V_\theta = 0 \rightarrow \boxed{\theta = 0 \text{ OR } \theta = \pi}$

$\rightarrow \underbrace{s = -mg \frac{\ell}{2} \cdot \frac{1}{\mu g}}_{<0}$

No condition since $s > 0$

$\downarrow$

$$k_2 s = \mu g \cos\theta$$

$$s = \frac{\mu g}{k_2} \cos\theta > 0 \text{ only for } \theta = 0.$$

From $(V_x, V_\theta, V_s) = (0, 0, 0)$ and by considering that $s > 0$, we obtain only 1 eq. configuration:

$$(x_1, \theta_1, s_1) = \left(0, 0, \frac{\mu g}{k_2}\right)$$

Is it stable?

$$\mathrm{Hess}\,V(x,\theta,s) = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & \cos\theta\left(\frac{mg\ell}{2}+\mu g s\right) & \mu g \sin\theta \\ 0 & \mu g\sin\theta & k_2 \end{pmatrix}$$

$$\mathrm{Hess}\,V\left(0,0,\frac{\mu g}{k_2}\right) = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & \frac{mg\ell}{2}+\frac{\mu^2 g^2}{k_2} & 0 \\ 0 & 0 & k_2 \end{pmatrix}$$

NON-DEGENERATE HESSIAN, POSITIVE DEFINITE $\Rightarrow$ EQ. IS STABLE!!

—x—

$\boxed{k_1 = 0}$

$$L = K_{TOT} - \underbrace{V(\theta, s)}_{\text{not dep. on } x!}$$

In such a case, we have 2 first integrals!

$E = K_{TOT} + V$ (as in the case $k_1 > 0$)

$$\frac{\partial L}{\partial \dot{x}} = \left(\frac{3}{2}M + m + \mu\right)\dot{x} + \frac{1}{2} m l \dot{\theta}\cos\theta + \mu \dot{s}\sin\theta + \mu s\dot{\theta}\cos\theta = \text{const}$$

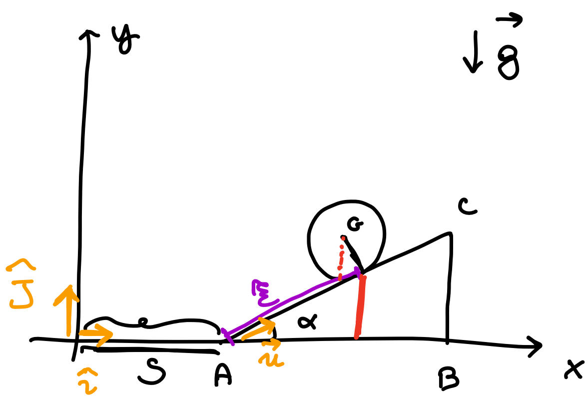
$\downarrow$ since x is a cyclic coordinate!

—x—

2 Triangle ABC M

Disc m, R

- Lagrangian of the system
- Determine first integrals.



$$K_M = \frac{1}{2} M \dot{s}^2$$

$= \dot{s}\,\hat{i} + \dot{\xi}\,(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

For the disc? we need to use Konig Theorem.

$$\vec{V} = \dot{s}\,\hat{i} + \dot{\xi}\,\hat{u} \implies |\vec{v}|^2 = \dot{s}^2 + \dot{\xi}^2 + 2\cos\alpha\,\dot{s}\dot{\xi}$$

$\downarrow$

velocity of the baricenter.

Since the disc purely rotates: $\dot{\xi} = -R\dot{\theta}$

$\downarrow$

($-$ depends on the chosen angle).

$$\frac{1}{2} I_G \dot{\theta}^2 = \frac{1}{2}\frac{m\cancel{R^2}}{2}\cdot\frac{\dot{\xi}^2}{\cancel{R^2}} = \frac{1}{2}\left(\frac{m}{2}\right)\dot{\xi}^2$$

$$K_{Disc} = \frac{1}{2} m\left[|\vec{v}|^2\right] + \frac{1}{2}\left(\frac{m}{2}\right)\dot{\xi}^2 = \frac{1}{2} m\left[\dot{s}^2 + \dot{\xi}^2 + 2\cos\alpha\,\dot{s}\dot{\xi}\right] + \frac{1}{2}\left(\frac{m}{2}\right)\dot{\xi}^2$$

$$K_{TOT} = \frac{1}{2}\left[(M+m)\dot{s}^2 + \frac{3}{2}m\dot{\xi}^2 + 2m\cos\alpha\,\dot{s}\dot{\xi}\right]$$

$$V = mg(\sin\alpha)\xi + \text{consts}$$

$$L = K_{TOT} - V$$

$L = L(\xi, \dot{\xi}, \dot{s}) \implies s$ is a cyclic coordinate!

$E = K_{TOT} + V$ first integral.

$\mathcal{R}(\xi, \dot{\xi}, P_s)$

$$\frac{\partial L}{\partial \dot{s}} = P_s = (M+m)\dot{s} + m\cos\alpha\,\dot{\xi} \rightarrow \dot{s} = \frac{P_s - m\cos\alpha\,\dot{\xi}}{(M+m)}$$

$=$ x-component of quantity of motion of the system ($\implies$ a conserved quantity).

Routh method

$L = L(q', \dot{q}', \dot{q}'', t) \qquad q = (q', \underbrace{q''}_{\text{cyclic coordinates.}})$

$q'' = (q_{m+1} - q_n)$ are cyclic.

while L depends explicitly on the other m first ones : $(q_1 - q_m)$.

In such a case:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_e} = 0 \qquad \forall\ e = m+1 - n \quad \text{and therefore}$$

the corresponding conj. momenta :

$$p_e = \frac{\partial L}{\partial \dot{q}_e} \qquad \forall\ e = m+1 - n \quad \text{are constants of motion.}$$

we intend to use these first integrals in order to write a "reduced" $\underbrace{\text{Lagrangian}}_{L_R}$, depending only on $(q', \dot{q}', t)$.

We consider the case of only 1- cyclic coordinate: q_n. The result is the following :

$$L_R(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{m-1}, t) :=$$

$$:= \left[L(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{m-1}, \dot{q}_n, t) - p_n \dot{q}_n \right]\Big|_{\dots}$$

$$\dots \dot{q}_n = u(q_1 - q_{m-1}, \dot{q}_1 - \dot{q}_{n-1}, t, \underbrace{p_n}_{\text{one parameter.}})$$

where we obtain :

$\dot{q}_n = u(\dots)$ by inverting

$$p_n = \frac{\partial L}{\partial \dot{q}_n}.$$

L_R is the reduced Lagrangian in the sense that the Lagrange eqs for the non cyclic coordinates are the same.

—— x ——