Definition: Let I be an interval.

$$\mathcal{C}(I) = \left\{ f : I \to \mathbb{R},\ f \text{ continuous on } I \right\}$$

$$\mathcal{C}^1(I) = \left\{ f : I \to \mathbb{R} \text{ s.t. differentiable on } I,\ f' \text{ is continuous on } I \right\}$$

- if I is open $I =]a, b[$ everything is clear
- if $I =]a, b]$ in b we mean left derivative
- if $I = [a, b[$ - - - - -

- - - -

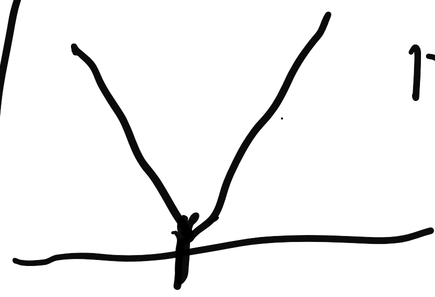
Theorem 1: $f : I \longrightarrow \mathbb{R}$

$x_0 \in \operatorname{int}(I)$ $\quad f'(x)$ exists in $]x_0, x_0+\varepsilon[$ (for some $\varepsilon > 0$)

Suppose $\lim\limits_{x \to x_0^+} f'(x) = l_1$

Then $l_1 = f'_+(x_0)$

Example



$|x|$

$0 = x_0$

Theorem 2: $f : D \longrightarrow \mathbb{R}$

$x_0 \in \operatorname{int}(I)$ $\quad f'(x)$ exists in $]x_0-\varepsilon, x_0[$ (for some $\varepsilon > 0$)

Suppose $\lim\limits_{x \to x_0^-} f'(x) = l_2$

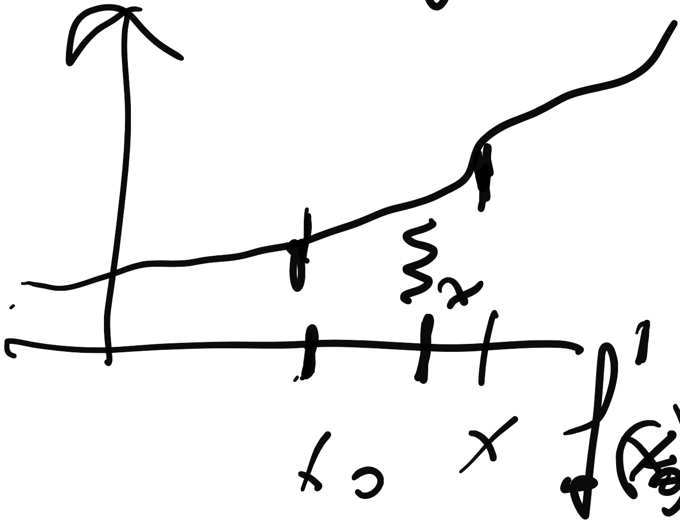
Then $l_2 = f'_-(x_0)$

Corollary: if $f'(x)$ exists in $]x_0-\varepsilon, x_0+\varepsilon[\setminus \{x_0\}$ and $\lim\limits_{x \to x_0} f'(x) = l$ then $\exists\, f'(x_0) = l$

Proof:

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

Apply Lagr. Th to $[x_0, x]$



$$\frac{f(x) - f(x_0)}{x - x_0} = f'(\xi_x)$$

$$f'_+(x_0) = \lim_{x \to x_0^+} \;\; '' \;\; = \lim_{x \to x_0^+} f'(\xi_x)$$

$$\boxed{\begin{array}{c} \xi_x \to x_0 \\ x \to x_0 \end{array}}$$

by hypothesi $''$ ℓ

Example: for any $n \in \mathbb{N}$, consider the function

$f_n(x) = x^n \sin\frac{1}{x}$. $f_n(0) = 0$

Study continuity, differentiability

$n = 0$ $f_0(x) = \begin{cases} \sin\frac{1}{x} & \forall x \neq 0 \\ 0 & x = 0 \end{cases}$

For $x \neq 0$

$\sin\frac{1}{x}$ is differentiable

$\Downarrow$

continuous

Is it continuous at $x = 0$?

No! $\lim_{x \to 0} \sin\frac{1}{x}$

$\quad\quad\quad\quad\quad\quad \text{''}$

$\quad\quad\quad\quad\quad f_0(x)$

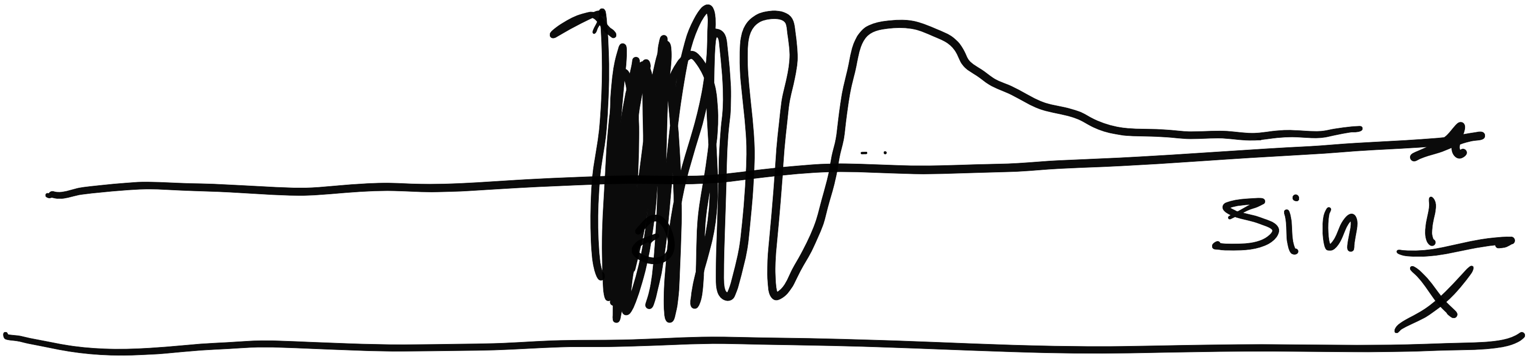
$x_k = \dfrac{1}{\frac{\pi}{2} + 2k\pi}$

$\hat{x}_k = \dfrac{1}{\frac{\pi}{4} + 2k\pi}$

$f_0(x_k) = 1 \to 1$

$f(\hat{x}_k) = \frac{\sqrt{2}}{2} \to \frac{\sqrt{2}}{2}$

no limit



$n = 1$

$$f_1(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$?

Continuous at $x=0$.

yes!

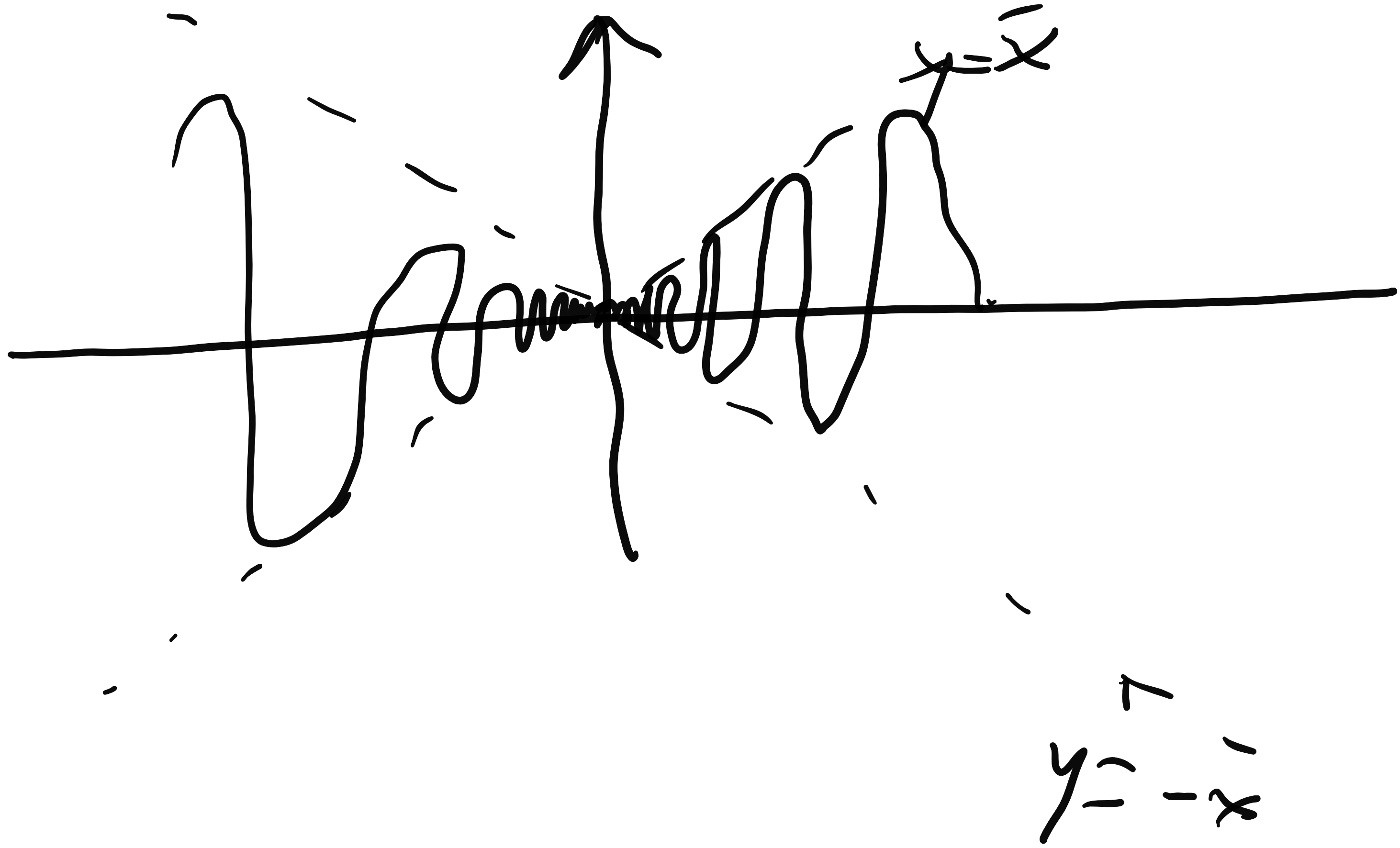
$$\lim_{x\to 0} f_1(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0 = f_1(0)$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \sin\frac{1}{x} + x\cos\frac{1}{x}\left(-\frac{1}{x^2}\right) =$$

$$\lim_{x\to 0}\left(\sin\frac{1}{x} - \cos\frac{1}{x}\cdot\frac{1}{x}\right)$$

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} =$$

$$\lim_{x\to 0}\frac{\cancel{x}\sin\frac{1}{x}}{\cancel{x}} =$$ does not exist.



$n = 2$ $\quad f_2$ is continuous

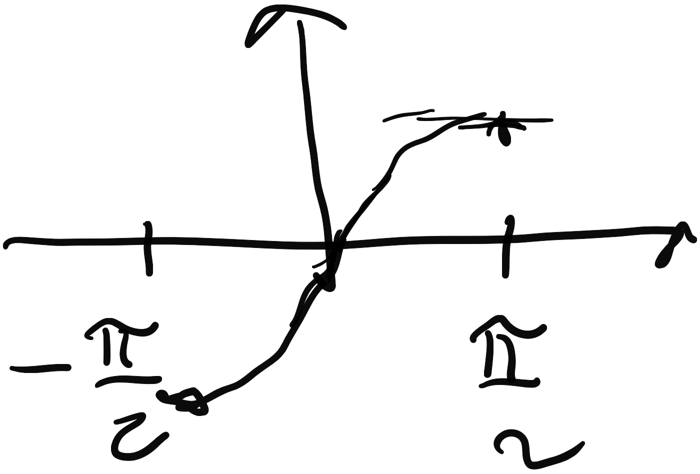
$$f_2'(x) = \underbrace{2x\sin\frac{1}{x} - \cancel{x^2}\cos\frac{1}{x}\cdot\frac{1}{\cancel{x^2}}}_{\text{no limit.}}$$

$$f_2'(0) = \lim_{x\to 0}\frac{x^{\cancel{2}}\sin\frac{1}{x}}{\cancel{x}} = 0$$

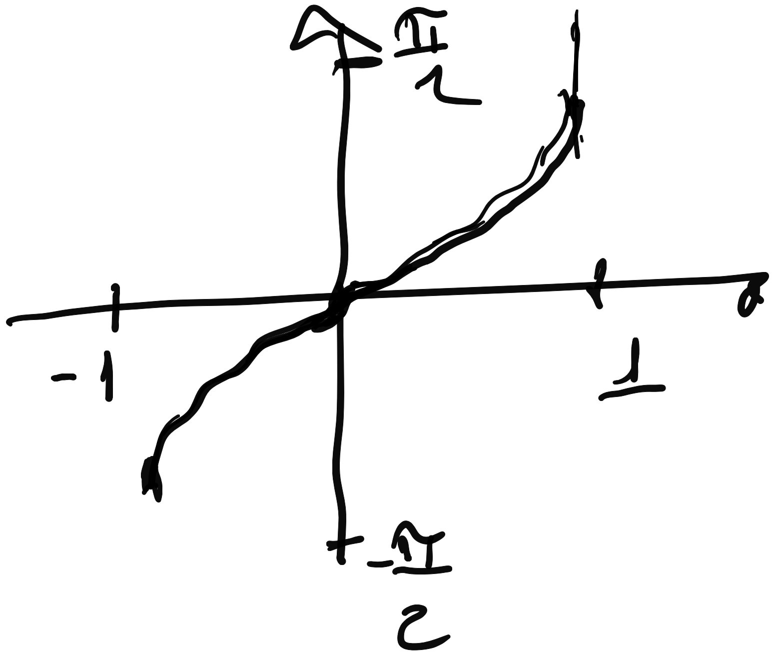
$n = 3$ $\quad f'(x) = \underbrace{3x^2\sin\frac{1}{x} - x^{\cancel{3}}\cos\frac{1}{x}\cdot\frac{1}{\cancel{x^2}}}_{\to 0}$

From Th. we can conclude that $f'(0) = 0$

We know $(\sin x)' = \cos x$



$(\arcsin y)' \overset{?}{=}$



$f : \underset{\text{open interval}}{I} \longrightarrow f(I)$

f bijective

$\Downarrow$

f is continuous

f^{-1} is continuous

Suppose f is differentiable

" you know that f^{-1} is differentiable

$f^{-1}(f(x)) = x \qquad \forall x \in I$

$$\left(f^{-1}(f(x))\right)' = \left(f^{-1}\right)'(f(x)) \cdot f'(x) = 1$$

$(f^{-1})'(f(x)) = 1/f'(x)$ that is $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(x))}$

DIFFERENTIATION OF THE INVERSE.

Theorem "Inverse map theorem"

$f : I \longrightarrow J = f(I)$

bijective, differentiable with $f'(x) \neq 0 \ \forall x$

Then the inverse map

$f^{-1} : J \longrightarrow I$

is differentiable and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Proof $(f^{-1})'(y) = \lim_{h \to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} =$

$$= \lim_{k \to 0} \frac{k}{f(x+k) - f(x)} = (*)$$

$$\boxed{x + k = f^{-1}(y+h)} \qquad x = f^{-1}(y)$$

$$\underbrace{k = f^{-1}(y+h) - x} = \underbrace{f^{-1}(y+h) - f^{-1}(y)}$$

$$y + h = f(x+k)$$

$$h = f(x+k) - y = f(x+k) - f(x)$$

$$(*) = \lim_{k \to 0} \frac{1}{\frac{f(x+k) - f(x)}{k}} = \frac{1}{f'(x)} =$$

$$= \frac{1}{f'(f^{-1}(y))}$$

- Find the derivative of arcsin, arccos, arcsinh, arccosh.
- Find the derivative of $\log(x)$
- Find the derivative of $\sqrt[3]{y} =$

$$\boxed{(\arcsin)'(x) = \quad \text{Th.}} \quad \frac{1}{(\sin)'(\arcsin(x))} =$$

$$= \frac{1}{\cos(\arcsin(x))} =$$

$$= \frac{1}{\sqrt{1-(\sin(\arcsin x))^2}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos)'(x) = \frac{1}{-\sin(\arccos x)} =$$

$$= \frac{1}{-\sqrt{1 - (\cos(\arccos(x)))^2}} =$$

$$= \frac{1}{-\sqrt{1 - x^2}}$$

$$\boxed{(\operatorname{arctg})'(x)} = \frac{1}{\frac{1}{\cos^2(\operatorname{arctg} x)}} =$$

$$\frac{1}{\cos^2(y)} = \frac{\sin^2(y) + \cos^2(y)}{\cos^2(y)} = 1 + \operatorname{tg}^2(x)$$

$$= \frac{1}{1 + \operatorname{tg}^2(\operatorname{arctg} x)} = \boxed{\frac{1}{1 + x^2}}$$