Theorem 1 (monotonicity and derivative's sign).

Let I be any interval, and let $f: I \longrightarrow \mathbb{R}$ a continuous function differentiable on $\text{int}(I)$.

1) f is increasing $\Longleftrightarrow$ $f'(x) \geq 0$ $\forall x \in \text{int}(I)$
2) f is decreasing $\Longleftrightarrow$ $f'(x) \leq 0$ $\forall x \in \text{int}(I)$
3) $f'(x) > 0$ $\forall x \in \text{int}(I)$ $\Longrightarrow$ f is strictly increasing
4) $f'(x) < 0$ $\forall x \in \text{int}(I)$ $\Longrightarrow$ f is strictly decreasing

Proof. Let us prove 1) "$\Longrightarrow$"

Since f is increasing $\dfrac{f(x+h)-f(x)}{h} \begin{cases} \geq 0 & \text{if } h>0 \\ \geq 0 & \text{if } h<0 \end{cases}$

$$\Longrightarrow \underbrace{\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}_{\geq 0} = f'(x)$$

$\Longrightarrow f'(x) \geq 0$ by Theorem of permanence of sign.

Let us prove 1) "$\Leftarrow$" Hypothesis: $f'(x) \geq 0$

$x_1 < x_2$ $x_1, x_2 \in I$. We have to prove

$f(x_1) \overset{?}{\leq} f(x_2)$. By Lagrange $\exists \xi \in]x_1, x_2[$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$$

$$f(x_2) - f(x_1) = \underbrace{f'(\xi)}_{\geq 0} \cdot \underbrace{(x_2 - x_1)}_{\geq 0} \geq 0$$

Let us prove 3)

$f'(x) > 0$ $\forall x \in \operatorname{int} I$. By Lagrange

$x_1 < x_2$

$$f(x_2) - f(x_1) = \underbrace{f'(\xi)}_{> 0} \cdot \underbrace{(x_2 - x_1)}_{> 0} > 0$$

Let us prove 4) same proof as 3)

Remark 1: What about the case when $\nexists\, f'(x_0)$ at a relative maximum or minimum point?

Examples: Study $f(x) = x^3 e^{-x}$ Maximal domain $= \mathbb{R}$

- $f(x) \neq f(-x)$ $f(x) \neq -f(-x)$

$$x^3 e^{-x} \stackrel{?}{=} -(-x)^3 e^{x}$$

$$x^3 e^{-x} = \cancel{x^3} e^{x}$$

- $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^3 e^{-x} = \lim_{x \to +\infty} \frac{x^3}{e^x} = 0$ → $y = 0$ is asymptote

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^3 e^{-x} = -\infty$$

- $f(x) \geq 0$ $x^3 \cdot \underbrace{e^{-x}}_{>0} \geq 0 \quad \forall x \geq 0$

$x^3 \geq 0$ $x \geq 0$

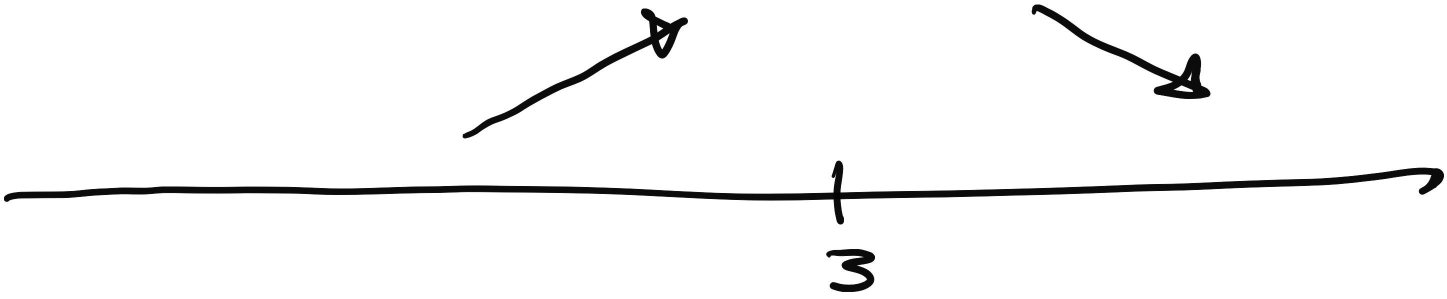
$x^3 \leq 0$ $x \leq 0$ $x^3 e^{-x} \leq 0 \quad \forall x \leq 0$

$$f'(x) = 3x^2 e^{-x} - e^{-x} x^3 = \underbrace{e^{-x} x^2 (3 - x)}_{} \geq 0$$

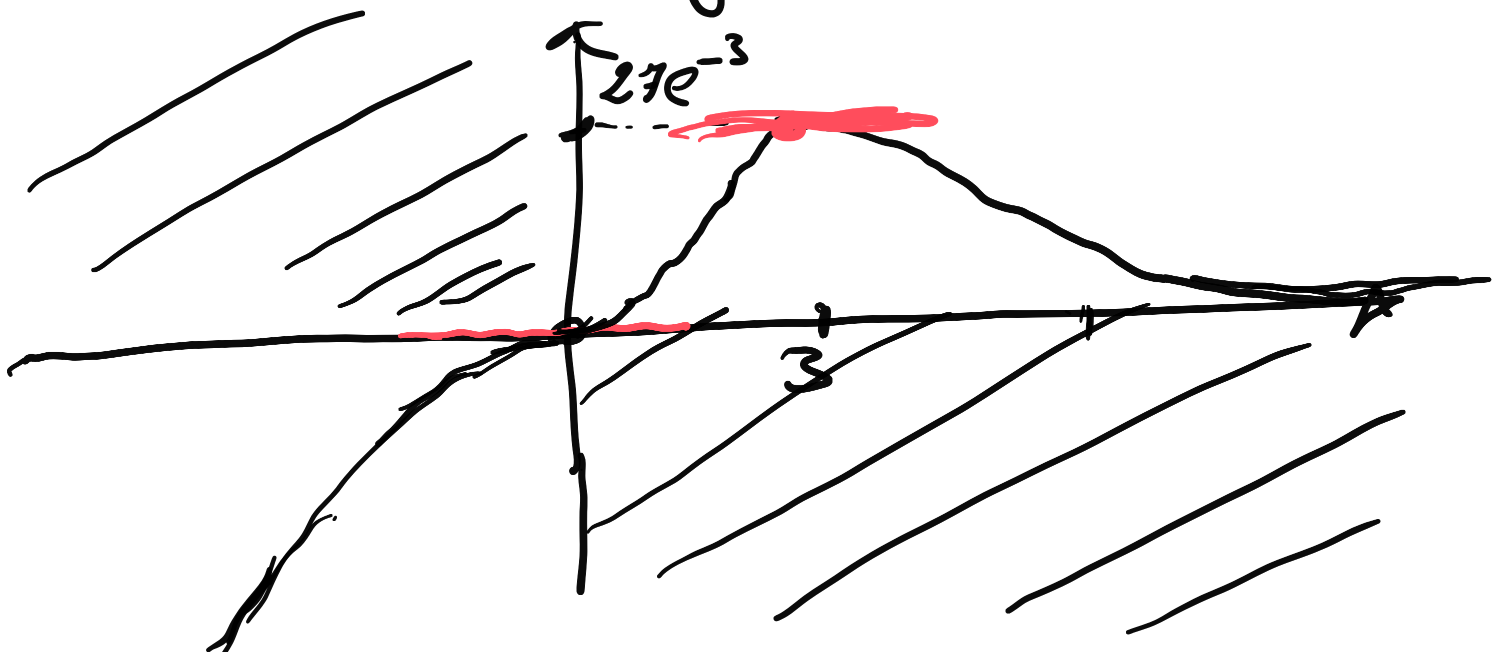
$$\underbrace{\begin{cases} e^{-x} x^2 \geq 0 \\ 3 - x \geq 0 \end{cases} \cup \begin{cases} e^{-x} x^2 \leq 0 \\ 3 - x \leq 0 \end{cases}}_{}$$

$$\Updownarrow$$

$$3 - x \geq 0 \qquad x \leq 3$$



$\Longrightarrow$ 3 is a global maximum point



The tangent at the maximum point $x = 3$

$y = \frac{27}{e^3}$

Asymptote at $-\infty$?

$$\lim_{x\to-\infty} \frac{f(x)}{x} = x^2 e^{-x} = +\infty$$

Study

$f(x) = |\sin x^4|$

Maximal domain $= \mathbb{R}$

$f(x) = |\sin x^4| \geq 0 \quad \forall \; x \in \mathbb{R}$

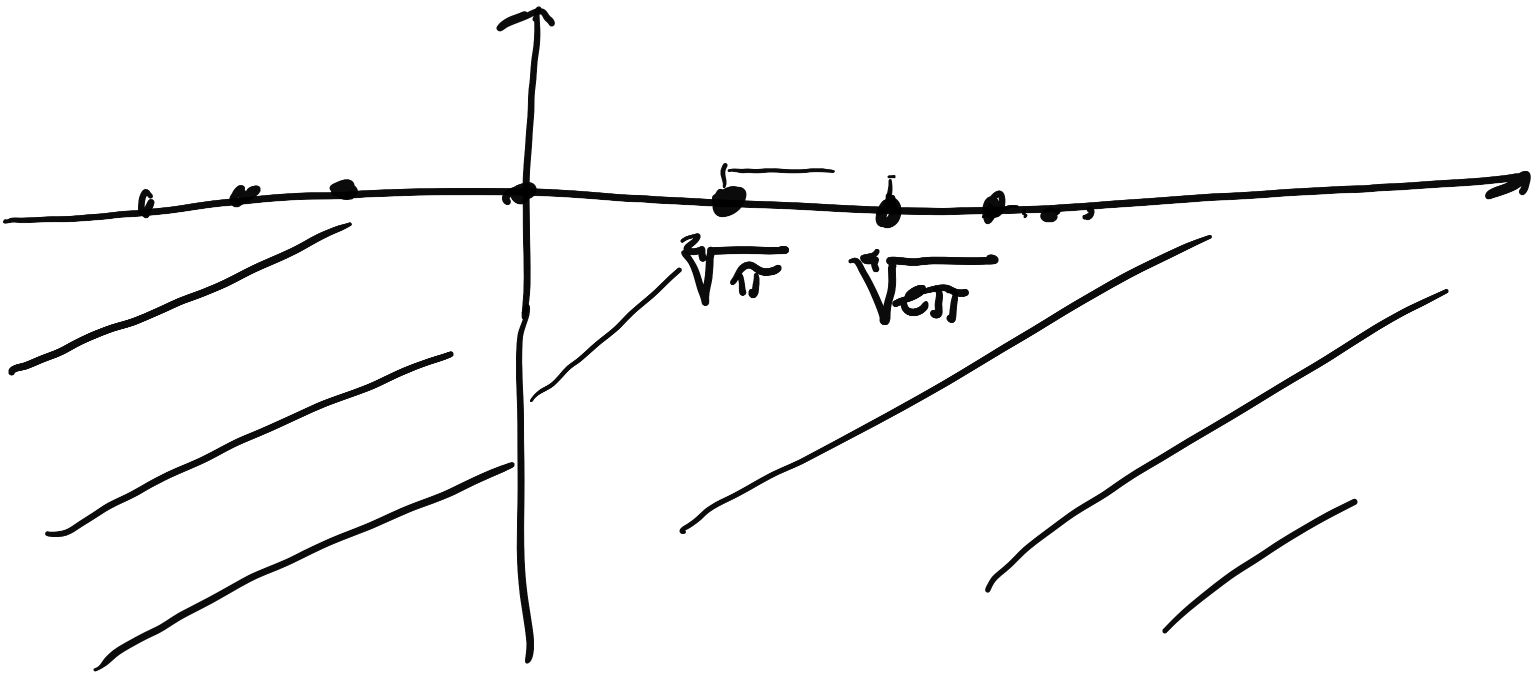
$f(x) = 0 \iff \sin x^4 = 0$

$\iff \forall \; x^4 = k\pi \quad k \in \mathbb{N}$

$x = \pm\sqrt[4]{k\pi} = (k\pi)^{\frac{1}{4}}$

$f(x) \stackrel{?}{=} f(-x)$ yes

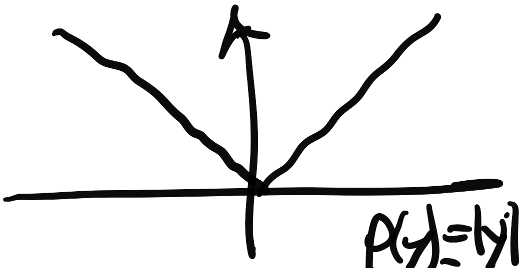
$\Rightarrow$ the function is even



$f'(x) = (|\sin x^4|)' = (g \circ h(x))'$

$h(x) = \sin x^4 \qquad g(y) = |y|$

$$(|y|)' = \begin{cases} 1 & y>0 \\ -1 & y<0 \end{cases}$$



$\nexists^{+} \quad y=0$

$$g'_+(0) = \lim_{y\to 0+} \frac{|y|}{y} = \lim_{y\to 0+}(1) = 1$$

$$g'_-(0) = \lim_{y\to 0-} = \lim_{y\to 0-}(-1) = -1$$

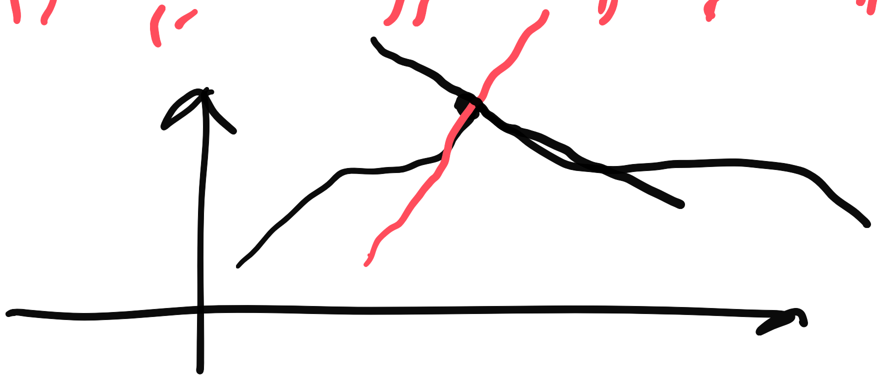
$g' = \text{sign}(y)$

Definition. $f: D \to \mathbb{R}$, $x_0 \in D$

$$f'_{\pm}(x_0) = \lim_{x\to x_0\pm} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0\pm} \frac{f(x_0+h)-f(x_0)}{h}$$

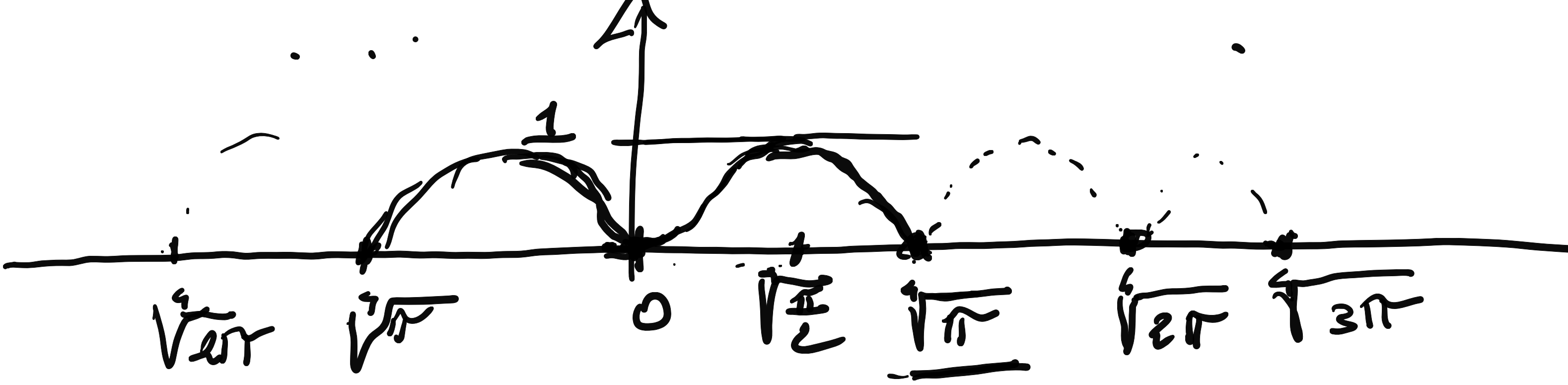
if it exists it is called the right derivative

'' '' '' '' '' '' the left derivative



$$f'(x) = (|\sin x^4|)' \underset{x\neq \pm\sqrt[4]{k\pi},\ k\in\mathbb{N}}{=} (g\circ h)' = g'(h(x))\cdot h'(x) =$$

$$= \{4\,\text{sign}(\sin(x^4))\cdot\cos(x^4)\cdot \underbrace{x^3}_{\geq 0}\} = f'(x)$$



$x \in]0, \sqrt[4]{\pi}[\quad \sin x^4 > 0$

$x \in]0, \sqrt[4]{\frac{\pi}{2}}] \quad \cos(x^4) \geq 0$

$x \in]\sqrt[4]{\frac{\pi}{2}}, \sqrt[4]{\pi}[\quad \cos(x^4) \leq 0$

$\Rightarrow f'(x) \geq 0$ on $]0, \sqrt[4]{\frac{\pi}{2}}[$

$f'(x) \leq 0$ on $]\sqrt[4]{\frac{\pi}{2}}, \sqrt[4]{\pi}[$

Question: $\lim\limits_{x \to 0^+} f'(x) = 0$

$\lim\limits_{x \to 0^-} f'(x) = 0$

$\lim\limits_{x \to \sqrt[4]{\pi}^-} f'(x) = -4\pi^{\frac{3}{4}} < 0$

Theorem: $f : D \longrightarrow \mathbb{R}$

$x_0 \in \mathrm{int}(I)$ $\quad f'(x)$ exists in $I \setminus \{x_0\}$

a neighbourhood I of x_0

Suppose $\lim\limits_{x \to x_0} f'(x) = \ell$

Then $\ell = f'(x_0)$

<u>Warning:</u> if $\lim\limits_{x\to x_0} f'(x)$ does NOT exist.

We CANNOT conclude that the function is not differ. at x_0.

<u>Example</u> $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$

$x\neq 0$

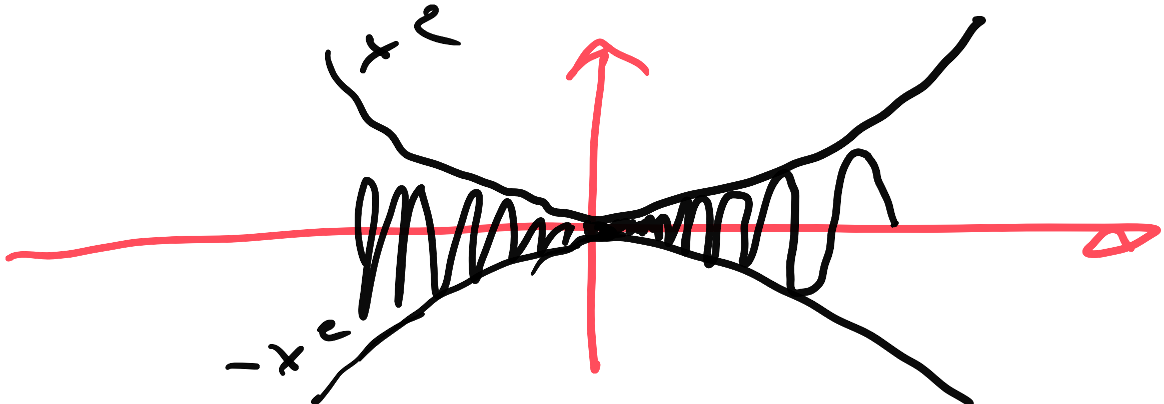
$$f'(x) = 2x\sin\left(\frac{1}{x}\right) + \cancel{x^2}\cos\frac{1}{x}\left(-\frac{1}{\cancel{x^2}}\right) =$$

$$= \overbrace{2x\sin\left(\frac{1}{x}\right)}^{\to 0} - \cos\frac{1}{x}$$

$\lim\limits_{x\to 0} f'(x)$ = does not exist.

but

$$\boxed{f'(0)} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^{\cancel{2}}\sin\frac{1}{x}}{\cancel{x}} \boxed{=0}$$



"Derivative of the inverse"

<u>Definition</u>

Spaces of functions $\mathcal{C}^0(I)$, $\mathcal{C}^1(I)$

Theorem (inverse mapping theorem)

Let $f - \mathcal{C}^1(I)$ with $f'(x) > 0$ (or $f'(x) < 0$) $\forall x \in I$.