Remind:

Weirstrass Theorem: $f: [a,b] \rightarrow \mathbb{R}$ continuos, then there exist a (global) maximum and a (global) minimum.

Fermat Theorem If $f: D \rightarrow \mathbb{R}$, if $\xi \in \mathrm{Int}(D)$, f differentiable at ξ, ξ a relative maximum or minimun, then

$$f'(\xi) = 0$$

Other fundamental theorems of differential calculus:

Theorem (Rolle):

Let $f : [a, b] \to \mathbb{R}$ be

- continuous on $[a, b]$
- differentiable on $]a, b[$
- such that $f(a) = f(b)$

Then there exists $\xi \in]a, b[$ such that

$$f'(\xi) = 0$$

Proof: By W. we know that there exists a minimum and a maximum.

I case) $\min f = f(x_m)$
$\max f = f(x_M)$
$f(x_m) = f(x_M)$

$\Longrightarrow f$ is constant

$\forall \xi \in]a,b[, \; f'(\xi) = 0$

II case $f(x_m) < f(x_M)$

x_m and x_M cannot be both end-points because $f(a) = f(b)$. x_m or x_M is $= \xi \in]a,b[\; \underset{\text{by Fermat}}{\Longrightarrow} f'(\xi) = 0$

q.e.d.

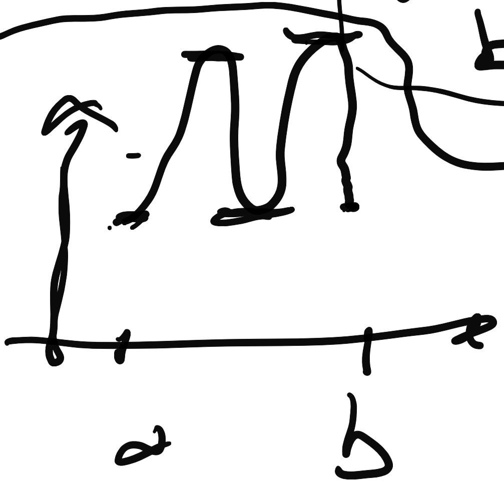
Theorem (Lagrange)

Let $f : [a,b] \longrightarrow \mathbb{R}$ be

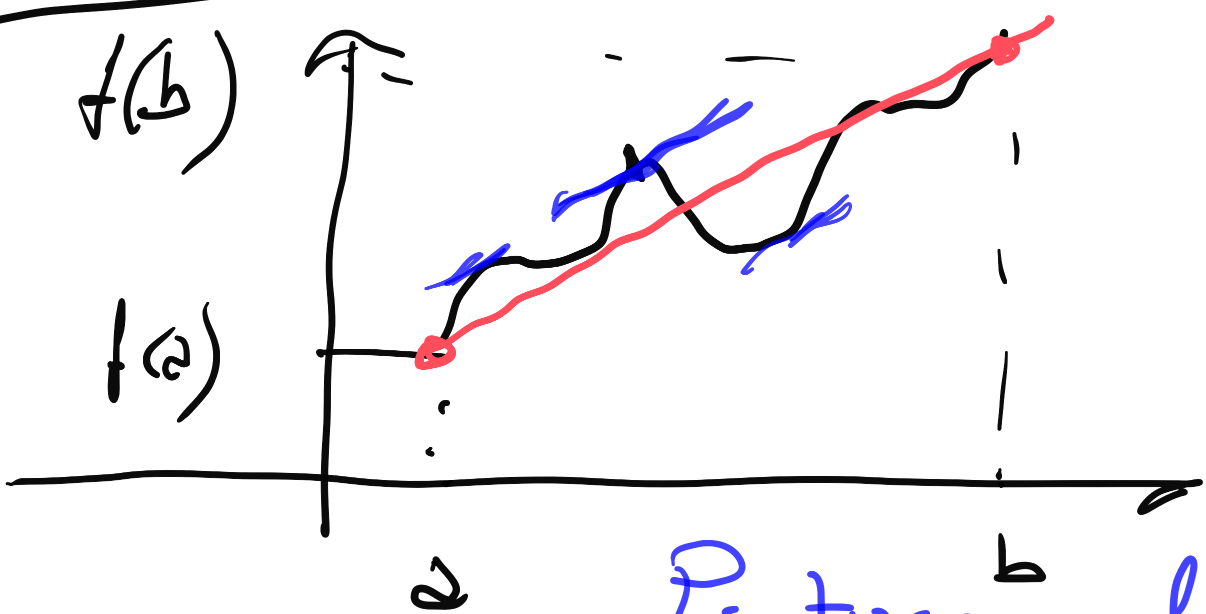
- continuous on $[a,b]$
- differentiable on $]a,b[$.

Then there exists $\xi \in]a,b[$ such that

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(\xi)}$$



Picture of Rolle Th



Picture of Lagrange th.

Proof: $g : [a,b] \longrightarrow \mathbb{R}$.

$$g(x) := f(x) - \frac{f(b)-f(a)}{b-a}\,x$$

$$g(a) = f(a) - \frac{f(b)-f(a)}{b-a}\,a =$$

$$= \frac{f(a)b - \cancel{f(a)a} - af(b) + \cancel{f(a)a}}{b-a} =$$

$$= \big(f(a)b - f(b)a\big)/(b-a)$$

$$g(b) = f(b) - \frac{f(b)-f(a)}{b-a}\cdot b =$$

$$= \frac{\cancel{f(b)b} - f(b)a - \cancel{f(b)b} + f(a)b}{b-a} =$$

$$= \big(f(a)b - f(b)a\big)/(b-a)$$

i.e. $\underline{g(a) = g(b)}$

g is continuous on $[a,b]$ (sum of continuous)

g is different. on $]a,b[$.

g verifies Rolle's hypotheses

$\xRightarrow{\text{Rolle's The}}$ $\exists\, \xi \in]a,b[$ s.t.

$$g'(\xi) = 0$$

$$\boxed{0} = g'(\xi) = \boxed{f'(\xi) - \frac{f(b) - f(a)}{b-a}}$$

$$\Downarrow$$

$$f'(\xi) = \frac{f(b) - f(a)}{b-a}$$

q.e.d.

Some consequences of Lagrange Theorem:

Theorem 1 (monotonicity and derivative's sign).

Let I be any interval, and let $f: I \longrightarrow \mathbb{R}$ a continuous function differentiable on $\text{int}(I)$

1) f is increasing $\Longleftrightarrow$ $f'(x) \geq 0 \ \forall x \in \text{int}(I)$
2) f is decreasing $\Longleftrightarrow$ $f'(x) \leq 0 \ \forall x \in \text{int}(I)$
3) $f'(x) > 0 \ \forall x \in \text{int}(I) \Longrightarrow f$ is strictly increasing
4) $f'(x) < 0 \ \forall x \in \text{int}(I) \Longrightarrow f$ is strictly decreasing

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Examples

$f: x \longrightarrow x^3$, $f: \mathbb{R} \to \mathbb{R}$

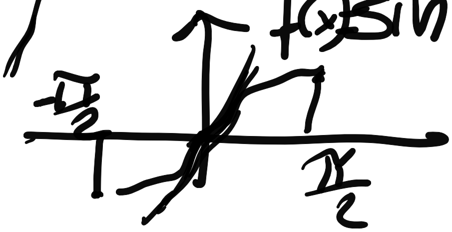
$f'(x) = 3x^2 = 0 \quad x = 0$

- $f(x) = x^\alpha$, $x \in [0, +\infty[$ $\quad \alpha \in \mathbb{R}$

$\boxed{\alpha > 0}$ $\quad f'(x) = \alpha x^{\alpha - 1} \geq 0 \quad \forall x \neq 0$

$\boxed{\alpha < 0}$ $\quad f'(x) = \alpha x^{\alpha - 1} < 0$

$f(x) = \sin x$



- $f(x) = \log x$ $\quad f' = \frac{1}{x} > 0$

$f:]0, +\infty[\to \mathbb{R}$ $\implies$ log is strictly increasi

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- $f(x) = e^x$ $\quad f'(x) = e^x > 0$

$f: \mathbb{R} \to \mathbb{R}$ $\implies$ f strict incr.

- $f(x) = x^3 - 4x^2 + 2x + 7$

$$f'(x) = 3x^2 - 8x + 2$$

$3x^2 - 8x + 2 \geq 0$ $\qquad x_{1,2} = \frac{8 \pm \sqrt{64-24}}{6}$

f': $+$ on $(-\infty, x_1)$, $-$ on (x_1, x_2), $+$ on $(x_2, +\infty)$

incr. ↗ dec. ↘ increasing ↗

- $f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x - (-1 e^{-x})}{2} =$$

$$= \frac{e^x + e^{-x}}{2} = \cosh(x) > 0$$

Exercise

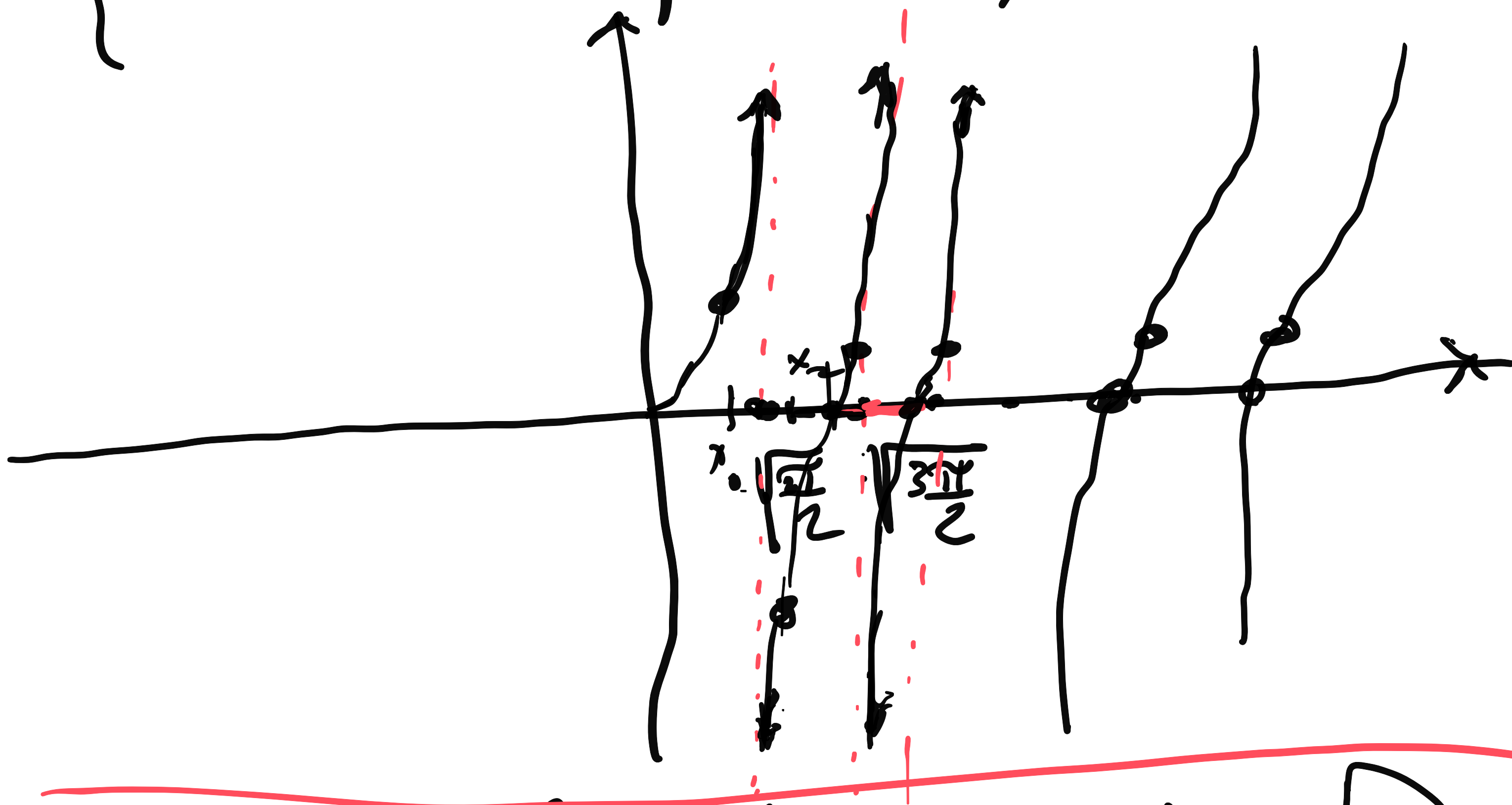
- Study the domain, simmetries, sign, limits, asymptotes, monotonicity of

$$f(x) = tg(x^2).$$

and draw a qualitative graph

Domain: $\{x \in \mathbb{R} : x^2 \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{N}\} =$

$= \{x \in \mathbb{R}\ \ x \neq \sqrt{\frac{\pi}{2} + k\pi}\} = D$



Even funct $f(x) = f(-x)$ $\forall x \in D$

Odd " $f(x) = -f(-x)$

$f(-x) = tg((-x)^2) = tg(x^2) = f(x)$

f is even $\Rightarrow$ Let us study it only on $D \cap [0, +\infty[$

$$\lim_{x \to \sqrt{\frac{\pi}{2}}^-} f(x) = \lim_{x \to \sqrt{\frac{\pi}{2}}^-} tg(x^2) \underset{\substack{x^2 \\ \shortparallel \\ y}}{=\!=} \lim_{y \to \frac{\pi}{2}^-} tg(y) = +\infty$$

$$\lim_{x \to \sqrt{\frac{\pi}{2}}^+} f(x) \ldots\ldots = -\infty$$

$$\lim_{x \to \sqrt{k\pi + \frac{\pi}{2}}^{\pm}} f(x) = \begin{cases} +\infty \\ -\infty \end{cases}$$

$$f'(x) = \frac{1}{\cos^2(x^2)} \cdot (2x)$$

for $x \in D \cap]0, +\infty[$ $\quad f'(x) > 0$

$\Rightarrow$ ~~f is increasing on $D \cap]0, +\infty[$~~

f is strictly increasing on every interval

$$\left]\sqrt{\frac{\pi}{2} + k\pi}, \sqrt{\frac{\pi}{2} + (k+1)\pi}\right[$$

$$\lim_{x \to +\infty} f(x) =$$

$$\lim_{k \to +\infty} f(\sqrt{k\pi}) = \lim_{k \to +\infty} tg(k\pi) = 0 \longrightarrow 0$$

$$\lim_{k\to+\infty} f\left(\sqrt{2k\pi + \frac{\pi}{4}}\right) = \lim_{k\to+\infty} tg\left(2k\pi + \frac{\pi}{2}\right) \equiv 1 \to 1$$

$$\Longrightarrow \quad \lim_{x\to+\infty} f(x) \text{ doesn't exist.}$$

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4) $f'(x) < 0 \ \forall x \in \mathrm{int}(I) \Longrightarrow f$ is strictly decreasing

Proof. 1) "$\Longrightarrow$"

$$\frac{f(x+h) - f(x)}{h} \begin{cases} \geq 0 & \text{if } h > 0 \\ \geq 0 & \text{if } h < 0 \end{cases}$$

$$\Longrightarrow \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\geq 0} = f'(x)$$

$$\Longrightarrow f'(x) \geq 0$$