2.6. Exercise

EXERCISE 2.6.1. *Show that* $\sqrt{3} \notin \mathbb{Q}$. *In general, can you prove that, for* $n \in \mathbb{N}$, $\sqrt{n} \notin \mathbb{Q}$ *unless* $n = m^2$ *for some* $m \in \mathbb{N}$?

EXERCISE 2.6.2. *Solve*

1. $\sqrt{x^2 + x - 1} \geqslant 1$.
2. $\sqrt{|x| + 1} > x - 1$.
3. $|x(x - 1)| \leqslant x - 1$.
4. $2^{2x} + 2^{x-1} \geqslant 3$.
5. $\dfrac{1}{2^x} - 2^x \geqslant 1$.
6. $2^x + 2^{-x} \leqslant 2$.
7. $\log_2(x^2) + \log_2(2x) \geqslant 0$.
8. $\log_2 |x + 1| + \log_2 |x - 3| < 1$.
9. $2^{|x-1|} < 4^{-x}$.

EXERCISE 2.6.3. *Find the domain of the following functions.*

1. $\sqrt{\log_2 x + \dfrac{2}{3}}$.
2. $\sqrt{2^{|x-3|} - 8} + \sqrt{3^{x^2+x+2} - 9}$.
3. $\log_{10}\left(2^x - \dfrac{1}{8}\right)$.
4. $\log_2\left(2 - \dfrac{\log_2(x - 2)}{\sqrt{1 + \log_2(x - 2)}}\right)$.
5. $2^{\sqrt{\frac{x^2-4}{2x^2-5x+3}}}$.
6. $\log_4 \sqrt{\dfrac{x + 1}{x - 1}}$.

EXERCISE 2.6.4. (1) *Let* $S := \left\{\frac{n}{\sqrt{n^2+1}} \ : \ n \in \mathbb{N},\ n \geqslant 1\right\}$, *show that* $\min S = \frac{1}{\sqrt{2}}$ *and* $\sup S = 1$. *What about* max?

(2) *Let* $S := \left\{\frac{n}{\sqrt{n^2-1}} \ : \ n \in \mathbb{Z},\ n \leqslant -2\right\}$, *show that* $\min S = -\frac{2}{\sqrt{3}}$ *and* $\sup S = -1$. *What about* max?

(3) *Let* $S := \{n - \sqrt{n - 3} \ : \ n \in \mathbb{N},\ n \geqslant 3\}$, *show that* $\min S = 3$ *and* $\sup S = +\infty$.

(4) *Let* $S := \left\{3 + \frac{\sqrt{2}}{\sqrt{n^2+7n-\sqrt{2}}} \ : \ n \in \mathbb{N},\ n > 0\right\}$, *show that* $\inf S = 3$ *and* $\max S = 4$. *What about* min?

(5) *Let* $S := \{-n + \sqrt{n - 2} \ : \ n \in \mathbb{N},\ n \geqslant 2\}$, *show that* $\max S = -2$ *and* $\inf S = -\infty$.

(6) *Let* $S := \left\{2 - \frac{\sqrt{2}}{\sqrt{n^2+2n-\sqrt{2}}} \ : \ n \in \mathbb{N},\ n > 0\right\}$, *show that* $\min S = 1$ *and* $\sup S = 2$. *What about* max?

(7) *Let* $S := \left\{\frac{\sqrt{n+1}}{\sqrt{n}+1} \ : \ n \in \mathbb{N},\ n \geqslant 1\right\}$, *show that* $\min S = \frac{\sqrt{2}}{2}$ *and* $\sup S = 1$. *What about* max?

Solutions:

Ex. 2.6.1. We prove that $\sqrt{3} \notin \mathbb{Q}$ by contradiction:

If it were $\sqrt{3} = \frac{m}{n}$ with m and n having no common factors, one would have

$$3 = \frac{m^2}{n^2} \iff 3n^2 = m^2 \quad (*)$$

So m^2 has a factor 3. Now this is possible only if m itself has a primefactor 3, i.e. $m = 3q$. Therefore by $(*)$ one has

$$3n^2 = m^2 = 9q^2 \iff$$

$$\iff n^2 = 3q^2 \iff n^2 \text{ has a factor } 3$$

$\iff$ n itself has a factor 3, i.e. $n = 3k$

Hence $m = 3q$ and $n = 3k$, which contradicts the assumption that m and n have no common factor.

We leave the second question as an (non trivial) exercise

Exercise 3.6.2

1) $\underbrace{\sqrt{x^2+x-1}}_{\geq 0} \geq \underset{0}{\underset{\vee\!\!/}{1}} \iff \begin{cases} x^2+x-1 \geq 0 \\ x^2+x-1 \geq 1 \end{cases}$

$\iff x^2+x-1 \geq 1 \iff x^2+x-2 \geq 0$

$\Delta = 1+8 = 9 \quad x_{1,2} = \dfrac{-1 \pm 3}{2} = \left\langle \begin{matrix} -2 \\ 1 \end{matrix} \right.$

Hence the set of solutions is is

$]-\infty, -2] \cup [1, +\infty[$

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2) $\sqrt{|x|+1} > x-1$

$\left\{\begin{matrix} |x|+1 \geq 0 \quad \forall x \in \mathbb{R} \\ x-1 < 0 \end{matrix}\right\} \bigcup \left\{\begin{matrix} |x|+1 \geq 0 \\ |x|+1 > (x-1)^2 \end{matrix}\right\}$

$\Updownarrow$

$\left\{ x < 1 \right\} \bigcup \left\{\begin{matrix} x \in \mathbb{R} \\ |x|+1 > x^2-2x+1 \end{matrix}\right\}$

$\Updownarrow$

$\left\{ x < 1 \right\} \bigcup \left\{\begin{matrix} x \geq 0 \\ x+1 > x^2-2x+1 \end{matrix}\right\} \bigcup \left\{\begin{matrix} x < 0 \\ -x+1 > x^2-2x+1 \end{matrix}\right\}$

$$\Updownarrow$$

$$]-\infty,1[\ \cup \begin{cases} x\geq 0 \\ x^2-3x<0 \end{cases} \cup \begin{cases} x<0 \\ x^2-x<0 \end{cases}$$

$$\Updownarrow$$

$$]-\infty,1[\ \cup \begin{cases} x\geq 0 \\ x<3 \end{cases} \cup \begin{cases} x<0 \\ x>1 \end{cases}$$

$$\Updownarrow$$

$$]-\infty,1[\ \cup\ [0,3[\ \cup\ \emptyset$$

$$\boxed{x\in\ ]-\infty,3[}\ \leftarrow \text{SOLUTION}$$

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3) $$|x(x-1)|\leq x-1$$

$$\Updownarrow$$

$$\begin{cases} x(x-1)\geq 0 \\ x^2-x\leq x-1 \end{cases} \cup \begin{cases} x(x-1)<0 \\ -x^2+x\leq x-1 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x\geq 1 \cup x\leq 0 \\ x^2-2x+1\leq 0 \end{cases} \cup \begin{cases} x\in\ ]0,1[\\ x^2-1\leq 0 \end{cases}$$

$$\begin{cases} x\in\ ]-\infty,0]\cup[1,+\infty[\\ (x-1)^2\leq 0 \end{cases} \cup \begin{cases} x\in\ ]0,1[\\ x\in\ ]-\infty,-1]\cup[1,+\infty[\end{cases}$$

$$\Updownarrow$$

$$\{x=1\} \cup \emptyset =$$

$$\boxed{x=1 \text{ is the only solution}}$$

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4) $2^x + 2^{x-1} \geq 3$

$$\Updownarrow$$

$$2^x + \frac{2^x}{2} \geq 3$$

$$\Updownarrow$$

$$\frac{3}{2} 2^x \geq 3$$

$$\Updownarrow$$

$$2^x \geq 2$$

$$\Updownarrow$$

$$\boxed{x \in [1, +\infty[}$$ — solution.

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5) $\frac{1}{2^x} - 2^x \geq 1$ Set $y = 2^x$ $(y>0)$

$$\frac{1}{y} - y \geq 1 \Leftrightarrow \frac{1-y^2-y}{y} \geq 0$$

$$\underset{(y>0)}{\Leftrightarrow} 1-y^2-y \geq 0 \Leftrightarrow y^2+y-1 \geq 0$$

$$y_{1,2} = \frac{-1 \mp \sqrt{5}}{2}.$$

$$\left\{\begin{array}{l} y \geq \frac{-1+\sqrt{5}}{2} \\ y > 0 \end{array}\right. \bigcup \left\{\begin{array}{l} y \leq \frac{-1-\sqrt{5}}{2} \\ y > 0 \end{array}\right.$$

$$y \in \left]\frac{-1+\sqrt{5}}{2}, +\infty\right[\bigcup \emptyset$$

$$\boxed{x \in \left]\log_2\left(\frac{-1+\sqrt{5}}{2}\right), +\infty\right[} \leftarrow \text{solution}$$

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6) $2^x + 2^{-x} \leq 2$ $\qquad 2^x = y$

$$y + \frac{1}{y} \leq 2 \iff \left\{\begin{array}{l} y > 0 \\ \frac{y^2+1-2y}{y} \leq 0 \end{array}\right.$$

$$\iff \left\{\begin{array}{l} y^2+1-2y \leq 0 \\ y > 0 \end{array}\right. \iff \left\{\begin{array}{l} (y-1)^2 \leq 0 \\ y > 0 \end{array}\right.$$

$$\iff y = 1 \iff \boxed{x = 0} \leftarrow \text{solution}$$

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7) $\log_2 x^2 + \log_2(2x) \geq 0$

$$\Updownarrow$$

$$\{2\log_2 x + \log_2 2 + \log_2 x \geq 0\} \cap \{x > 0\}$$

$$\Updownarrow$$

$$\{3\log_2 x + 1 \geq 0\} \cap \left]0, +\infty\right[$$

$$\Updownarrow$$

$$\begin{cases} \log_2 x \geq -\frac{1}{3} \\ x > 0 \end{cases} \Longleftrightarrow \boxed{x \in \left[\frac{1}{\sqrt[3]{2}}, +\infty\right[}$$

solution

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8) $\log_2|x+1| + \log_2|x-3| < 1$

$$\Updownarrow$$

$$\begin{cases} |x+1| > 0 \\ |x-3| > 0 \\ \log_2(|(x+1)(x-3)|) < 1 \end{cases} \Longleftrightarrow \begin{cases} x \neq -1 \\ x \neq 3 \\ |(x+1)(x-3)| < 2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x \neq -1 \\ x \neq 3 \\ x \in]-\infty, -1[\cup]3, +\infty[\\ x^2 - 2x - 3 < 2 \end{cases} \cup \begin{cases} x \neq -1 \\ x \neq 3 \\ x \in]-1, 3[\\ -x^2 + 2x + 3 < 2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x \in]-\infty, -1[\cup]3, +\infty[\\ x^2 - 2x - 5 < 0 \end{cases} \cup \begin{cases} x \in]-1, 3[\\ x^2 - 2x - 1 > 0 \end{cases}$$

$$\Updownarrow$$

$$x \in \left(]-\infty, -1[\cup]3, +\infty[\right) \cap \left(]1-\sqrt{6}, 1+\sqrt{6}[\right) \cup \left(]-1, 3[\cap]1-\sqrt{2}, 1+\sqrt{2}[\right)$$

$$\Updownarrow$$

$$\boxed{x \in]1-\sqrt{6}, -1[\cup]3, 1+\sqrt{6}[\cup]1-\sqrt{2}, 1+\sqrt{2}[}$$

SOLUTION

9) $2^{|x-1|} < 4^{-x} \iff 2^{|x-1|} < 2^{-2x} \iff$

$\iff |x-1| < -2x$

$\iff \begin{cases} x-1 \leq 0 \\ 1-x < -2x \end{cases} \cup \begin{cases} x-1 > 0 \\ x-1 < -2x \end{cases}$

$\iff \begin{cases} x \leq 1 \\ x < -1 \end{cases} \cup \begin{cases} x > 1 \\ x < \frac{1}{3} \end{cases}$

$\Updownarrow$

$x \in]-\infty, -1[\cup \emptyset$

$\Updownarrow$

$\boxed{x \in]-\infty, -1[}$

solution