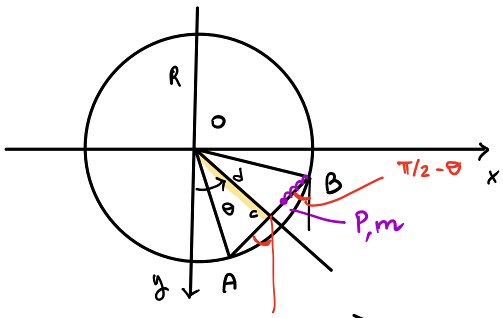
Lesson 22 - 17/11/2022

EX 1



S = length of spring
θ = angle
R radius
K spring constant
$|AB| = 2\ell$, negl. mass

- Lagrangian (with V_{el})
- First integral(s)

$d = \sqrt{R^2 - \ell^2} \Rightarrow \vec{OC} = (d \sin\theta, d\cos\theta)$

$\vec{OB} = (d\sin\theta + \ell \sin(\pi/2 - \theta), d\cos\theta - \ell\cos(\pi/2-\theta))$

$= (d\sin\theta + \ell\cos\theta, d\cos\theta - \ell\sin\theta)$

Finally

$$\boxed{\vec{OP} = (d\sin\theta + \ell\cos\theta - s\cos\theta, d\cos\theta - \ell\sin\theta + s\sin\theta)}$$

Therefore

$\vec{v}_P = (d\dot\theta\cos\theta - \ell\dot\theta\sin\theta + s\dot\theta\sin\theta - \dot s\cos\theta,$
$-d\dot\theta\sin\theta - \ell\dot\theta\cos\theta + s\dot\theta\cos\theta + \dot s\sin\theta)$

Hence

$|\vec{v}_P|^2 = \dots = d^2\dot\theta^2 + \ell^2\dot\theta^2 + \dot s^2 + s^2\dot\theta^2 - 2\ell s\dot\theta^2 - 2d\dot\theta\dot s$

$K = \frac{1}{2}m|\vec{v}_P|^2 = \frac{1}{2}m\left[d^2\dot\theta^2 + \ell^2\dot\theta^2 + \dot s^2 + s^2\dot\theta^2 - 2\ell s\dot\theta^2 - 2d\dot\theta\dot s\right]$

$V = V_{el} = \frac{1}{2}ks^2$

$L = K - V_{el} = L(s, \dot s, \dot\theta)$

$\Rightarrow \theta$ is a cyclic coordinate $\Rightarrow$ the corresponding conj. momentum is a conserved quantity!

It's the conservation of the angular momentum. $m(\vec{OP} \wedge \vec{v}_P)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \underbrace{\frac{\partial L}{\partial\theta}}_{\equiv 0} = 0 \Rightarrow \boxed{\frac{\partial L}{\partial\dot\theta} = \dots \text{ is a first integral.}}$

Moreover $E = K + V_{el}$ is a first integral.

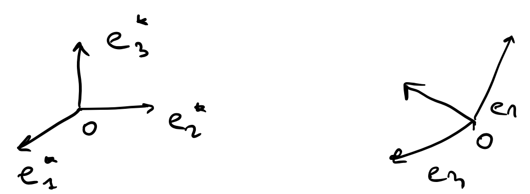
Moreover, the other Lagr. eq, gives:

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot s}\right) - \frac{\partial L}{\partial s} = 0 \iff \frac{d}{dt}\left(m\dot s - md\dot\theta\right) - \left(ms\dot\theta^2 - m\ell\dot\theta^2 - ks\right) = 0$

- Towards rigid motions... The fundamental formula of rigid motions.
- L-D theorem (with proof).

$\boxed{\text{Towards rigid motions...}}$

Suppose to know the motion of a point respect to a fixed orthonormal base (O, e_1^*, e_2^*, e_3^*) [$(1,0,0)$, $(0,1,0)$, $(0,0,1)$ for example].
We want to know the motion of the point wrt a orthonormal base in motion (O, e_1, e_2, e_3).



We remark that $\forall\, t \in \mathbb{R}$ $\exists\, A_t$ invertible matrix such that

$$e_i(t) = A_t e_i^* \qquad (1) \qquad \forall\, i = 1,2,3.$$

In particular $(A_t)_{ij} = e_i^* \cdot e_j(t)$ (2) $\forall\, i,j = 1,2,3$

that is, A_t is the matrix whose j-column is given by components of the vector e_j with respect to the base (e_1^*, e_2^*, e_3^*).

Prop When the two basis are orthonormal, that is

$$e_i \cdot e_j = e_i^* \cdot e_j^* = \delta_{ij} \qquad \forall\, i,j = 1,2,3$$

then A_t is <u>orthogonal</u>, that is $A_t^{-1} = A_t^T$.

Equivalently: $A_t \in O(3)$.

Proof In the proof we omit the dependence on $t \in \mathbb{R}$.

$(A)_{ij} = e_i^* \cdot e_j$. Therefore:

$$(A^T A)_{ij} = \sum_{k=1}^{3} (A^T)_{ik} (A)_{kj} =$$

$$= \sum_{k=1}^{3} (A)_{ki} (A)_{kj} = \sum_{k=1}^{3} (e_k^* \cdot e_i)(e_k^* \cdot e_j) =$$

$$= e_i \cdot e_j = \delta_{ij}$$

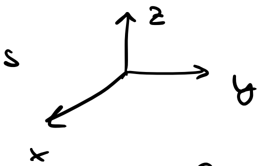
$$\Rightarrow A^T A = \mathbb{1} \Rightarrow A^T = A^{-1} \quad (\forall t \in \mathbb{R}) \qquad \boxtimes$$

As a consequence:

$A_t^T A_t = \mathbb{1}$, we obtain $\det(A_t^T A_t) =$

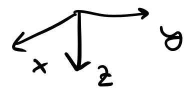
$= \det(A_t^2) = 1 \Rightarrow \det(A_t) = 1$ or $\det(A_t) = -1$.

In particular, if the base in motion is



then the matrix A_t has det 1 $\forall t \in \mathbb{R}$. So A_t is a rotation: $A_t \in SO(3)$.

If the base in motion is



then the matrix A_t has det -1 $\forall t \in \mathbb{R}$. So A_t is a reflection.

$$e_i = A_t e_i^* \Rightarrow$$

$$\dot{e}_i(t) = \dot{A}_t e_i^* = \dot{A}_t \underbrace{A_t^{-1} e_i}_{(1)} = \boxed{\dot{A}_t A_t^T} e_i(t)$$

$$A_t^{-1} = A_t^T$$

(since $A_t \in O(3)$)

Prop $t \mapsto A_t \in SO(3)$ differentiable.

Then $\dot{A}_t A_t^T$ is antisimmetric, that:

$$(\dot{A}_t A_t^T)^T = -(\dot{A}_t A_t^T)$$

Equivalently: $\dot{A}_t A_t^T \in \text{Skew}(3) \quad \forall t \in \mathbb{R}$.

Proof $A_t^{-1} = A_t^T \Rightarrow \mathbb{1} = A_t A_t^T$

Now, derive wrt $t \in \mathbb{R}$.

$$\mathbb{1} = A_t A_t^T \Rightarrow \mathbb{O} = \frac{d}{dt}(A_t A_t^T) =$$

$$= \dot{A}_t A_t^T + A_t \dot{A}_t^T = \dot{A}_t A_t^T + (\dot{A}_t A_t^T)^T \Rightarrow$$

$$\dot{A}_t A_t^T = -(\dot{A}_t A_t^T)^T \qquad \boxtimes$$

Prop $\exists$ isomorphism of vector spaces between $\mathbb{R}^3$ and $Skew(3)$. The isomorphism is defined as follows:

$$\mathbb{R}^3 \ni \vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \longmapsto \widehat{\vec{\omega}} = \underbrace{\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}}_{\in\, Skew(3)}$$

Proof The map $\widehat{\ }: \mathbb{R}^3 \longrightarrow Skew(3)$ is linear and invertible. $\boxtimes$

In our case... $\dot{A}_t A_t^T \in Skew(3)$

$\forall t \in \mathbb{R}$. Then $\exists\ \vec{\omega}(t) \in \mathbb{R}^3$ s.t.

$$\widehat{\vec{\omega}}(t) = \dot{A}_t A_t^T \in Skew(3) \quad \forall t \in \mathbb{R}.$$

Therefore:

- $$\dot{e}_i(t) = \underbrace{\dot{A}_t A_t^T}_{\in\, Skew(3)} e_i(t) =$$

$$= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} e_i(t)$$

$$= \vec{\omega}(t) \wedge e_i(t).$$

$$\Downarrow$$

How to express $(\dot{e}_1(t), \dot{e}_2(t), \dot{e}_3(t))$ with respect to $(e_1(t), e_2(t), e_3(t))$?

By means a unique vector: $\vec{\omega}(t)$

$$\boxed{\dot{e}_i(t) = \vec{\omega}(t) \wedge e_i(t) \qquad \forall i = 1, 2, 3 \quad \forall t}$$

$\swarrow$

Poisson formula

Sometimes is useful to write $\vec{\omega}(t)$ with respect to $\dot{e}_1(t)$, $e_i(t)$ ...

$$e_i \wedge \dot{e}_i = e_i \wedge (\omega \wedge e_i) =$$

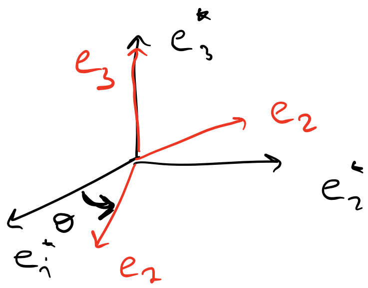
$$= \omega - (e_i \cdot \omega) e_i \quad \Rightarrow$$

$$\sum_{i=1}^{3} (e_i \wedge \dot{e}_i) = \sum_{i=1}^{3} \omega - \underbrace{(e_i \cdot \omega) e_i}_{\omega}$$

$$= 3\omega - \omega = 2\omega$$

$$\Rightarrow \boxed{\vec{\omega} = \frac{1}{2}\sum_{i=1}^{3} (e_i \wedge \dot{e}_i)}$$

The simplest example (Rotation on a plane)



$\theta = \theta(t)$ = angle between e_1^* and e_1 (counterclockwise)

$$A_t = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \dot{A}_t A_t^T = \dots = \begin{pmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \vec{\omega}(t) = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}.$$

Rigid motions

S = system of points

$(O^*, e_1^*, e_2^*, e_3^*)$

$(O(t), e_1(t), e_2(t), e_3(t))$

fixed base & base in motion respectively.

Def The movement of S is RIGID if the distance of each pair of points of S remains fixed.

In such a case, there is a special base in motion we can use. The one whose components of vectors of points in S remain constant (SOLIDALE). That is:

$$OP_i(t) = \underbrace{x_1}\, e_1(t) + \underbrace{x_2}\, e_2(t) + \underbrace{x_3}\, e_3(t)$$

$$\shortparallel$$

$$P_i - O(t)$$

$$OP_J(t) = y_1 e_1(t) + y_2 e_2(t) + y_3 e_3(t)$$

$$\underset{\displaystyle V_i}{\overset{\shortparallel}{\dot{OP}_i(t)}} - \underset{\displaystyle V_J}{\overset{\shortparallel}{\dot{OP}_J(t)}} =$$

$$= x_1 \dot{e}_1(t) + x_2 \dot{e}_2(t) + x_3 \dot{e}_3(t) -$$

$$- [y_1 \dot{e}_1(t) + y_2 \dot{e}_2(t) + y_3 \dot{e}_3(t)]$$

$$= (x_1 - y_1)\vec{\omega} \wedge e_1 + (x_2 - y_2)\vec{\omega} \wedge e_2 +$$

$$+ (x_3 - y_3)\vec{\omega} \wedge e_3 = \vec{\omega} \wedge \underbrace{P_J P_i}_{1}$$

$$\Downarrow \qquad\qquad\qquad \downarrow$$

$$P_i - P_J$$

$$\boxed{\vec{V}_i = \vec{V}_J + \vec{\omega} \wedge P_J P_i}$$

$$\begin{array}{c} \| \\ P_i - P_J \end{array}$$

→ Fundamental formula of rigid motions.

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Lagrange - Dirichlet Theorem

Dynamical system with ideal constraints, not depending on time (fixed), not depending on velocities (holonomic).

$$Q_h(q, \dot{q}) = Q_h^1(q) + Q_h^2(q, \dot{q})$$

such that:

i) $Q_h^1(q) = -\dfrac{\partial V}{\partial q_h}$ (positional, conservative forces).

ii) $Q_h^2(q, \dot{q})$ s.t. $Q_h^2(q^*, 0) \equiv 0$

and $\sum_{h=1}^{N} Q_h^2(q, \dot{q})\, \dot{q}_h \leq 0$.

Let suppose that $\dfrac{\partial V}{\partial q_h}(q^*) \equiv 0 \quad \forall h$

[$(q^*, 0)$ is an equilibrium]

IF q^* is a strict minimum of V
THEN $(q^*, 0)$ is STABLE.

— x —

1) $Q_n^2(q, \dot q)$ is called GYROSCOPIC
if $Q_n^2(q, \dot q)\, \dot q_n = 0$ (Coriolis force....)

2) $Q_n^2(q, \dot q)$ is called DISSIPATIVE
if $Q_n^2(q, \dot q)\, \dot q_n < 0$ (Viscous force...)