MATH4U - 17/11/2022

Thursday, 17 November 2022 11:36

1) REVIEW EXERCISES FROM PREVIOUS EXAMS
2) CONTINUOUS AND DERIVABLE FUNCTIONS
3) STUDY OF A FUNCTION

1) $\operatorname{Im}\left(\frac{1}{z}\right) \geq \frac{\operatorname{Im}(z^2 - \bar{z}^2)}{|z|^2}$

$z = x + iy$

$\operatorname{Im}\left(\frac{1}{x+iy}\right) \geq \frac{\operatorname{Im}\left((x+iy)^2 - (x-iy)^2\right)}{x^2+y^2}$

$\frac{1}{x+iy} \approx \underbrace{a}_{Re} + i\underbrace{b}_{Im} \Rightarrow \frac{1}{(x+iy)}\frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\underbrace{\frac{-y}{x^2+y^2}}_{\operatorname{Im}\left(\frac{1}{z}\right)}$

$x^2 - y^2 + 2ixy - [x^2 - y^2 - 2ixy] = \cancel{x^2} \cancel{-y^2} + 2ixy \cancel{-x^2} \cancel{+y^2} + 2ixy$

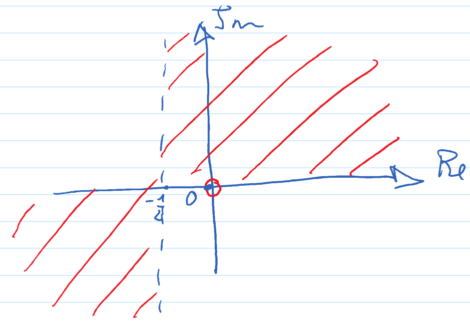
$= \underbrace{4ixy}_{\to 4xy} \longrightarrow 0 + i\underbrace{(4xy)}$

$\Rightarrow \frac{-y}{x^2+y^2} \geq \frac{4xy}{x^2+y^2}$ $\quad x^2+y^2 \to \geq 0$

$\begin{cases} x^2+y^2 \neq 0 \\ -y \geq 4xy \end{cases} \qquad (4x+1)y \geq 0$

$\nearrow$ $y \geq 0 \lor x \geq -\frac{1}{4}$

$\searrow$ $y \leq 0 \lor x \leq -\frac{1}{4}$



2) $\operatorname{Im}(\bar{z}^2 z - z^2 \bar{z}) = 4\operatorname{Re}(iz)$

$z = x + iy$

$\operatorname{Im}(\underbrace{\bar{z}z}_{|z|^2}(\bar{z} - z)) = \operatorname{Im}\left[(x^2+y^2)(\cancel{x} - iy - (\cancel{x} + iy))\right] = \operatorname{Im}\left[(x^2+y^2)(-2iy)\right]$

$= -2y(x^2+y^2)$

$4\operatorname{Re}(iz) = 4\operatorname{Re}[i(x+iy)] = 4\operatorname{Re}[ix - y] = -4y$

$\Rightarrow -2y(x^2+y^2) = -4y$

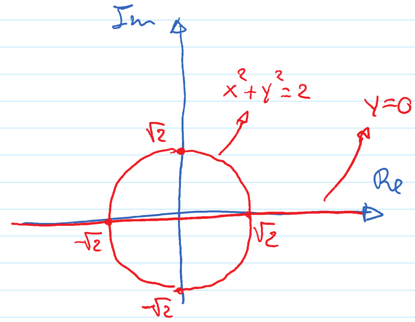
Im

$$\Longrightarrow \quad -2y(x^2+y^2) = -4y$$

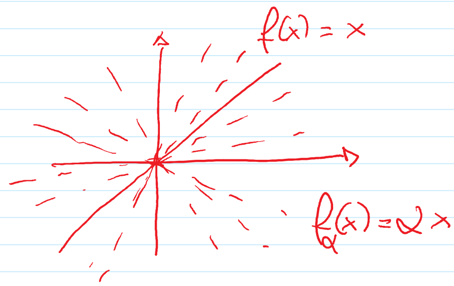
$$y = 0 \quad \forall x \in \mathbb{R}$$

v

$$\begin{cases} y \neq 0 \\ x^2+y^2 = 2 \end{cases} \quad \rightarrow \text{circle: center}(0,0) \ \text{radius } \sqrt{2}$$



3) $$\lim_{x \to +\infty} \frac{\left(\cosh \frac{1}{x} - 1\right)^2 - e^{-x}}{\left(\log(2+x) - \log x + \frac{2\alpha}{x}\right)^2} \qquad \alpha \in \mathbb{R}$$



N: $\left(\cosh \frac{1}{x} - 1\right)^2 - e^{-x}$

$\quad \hookrightarrow \frac{1}{x} \to 0$ for $x \to +\infty$

$$\left[\cancel{1} + \underbrace{\frac{1}{2}\left(\frac{1}{x}\right)^2} + o\left(\frac{1}{x^2}\right) \cancel{-1}\right]^2 - e^{-x}$$

$$\left[\frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right]^2 = \underbrace{\left(\frac{1}{2x^2}\right)^2}_{\sim \frac{1}{x^4}} + \underbrace{\left(o\left(\frac{1}{x^2}\right)\right)^2}_{\sim o\left(\frac{1}{x^4}\right)} + \underbrace{2\left(\frac{1}{2x^2}\right) o\left(\frac{1}{x^2}\right)}_{\sim o\left(\frac{1}{x^4}\right)}$$

$$\frac{1}{4x^4} + o\left(\frac{1}{x^4}\right) - e^{-x} \rightarrow e^{\overbrace{-x}^{\to -\infty}} \to 0 \qquad e^{-x} = o\left(\frac{1}{x^4}\right)$$

$$\frac{1}{4x^4} + o\left(\frac{1}{x^4}\right)$$

D: $$\left[\log(2+x) - \log x + \frac{2\alpha}{x}\right]^2$$

$\log(2+x) \to +\infty$, $\log x \to +\infty$: $[+\infty - \infty]$; $\frac{2\alpha}{x} \to 0$

$$f(x) - \frac{f(x)^2}{2}$$

$$\log\left(\frac{2+x}{x}\right) = \log\left(1 + \frac{2}{x}\right) = \frac{2}{x} - \frac{1}{2}\left(\frac{2}{x}\right)^2 + o\left(\frac{1}{x^2}\right) \qquad \frac{2}{x} \to 0$$

$$\left[\frac{2}{x} - \frac{2}{x^2} + \frac{2\alpha}{x} + o\left(\frac{1}{x^2}\right)\right]^2 = \left[(1+\alpha)\frac{2}{x} - \frac{2}{x^2} + o\left(\frac{1}{x^2}\right)\right]^2$$

$$\frac{N}{D} \quad \lim_{x \to +\infty} \frac{\frac{1}{4x^4} + o\left(\frac{1}{x^4}\right)}{\left[(1+\alpha)\frac{2}{x} - \frac{2}{x^2} + o\left(\frac{1}{x^2}\right)\right]^2}$$

$\alpha \neq -1$
$$\lim_{x \to +\infty} \frac{\frac{1}{4x^4} + o\left(\frac{1}{x^4}\right)}{\left[(1+\alpha)\frac{2}{x} + o\left(\frac{1}{x}\right)\right]^2} = \lim_{x \to +\infty} \frac{\frac{1}{4x^4} + o\left(\frac{1}{x^4}\right)}{(1+\alpha)^2\frac{4}{x^2} + o\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x \to +\infty} \frac{1}{16(1+\alpha)^2}\frac{1}{x^2} = 0$$

$$= \lim_{x \to +\infty} \frac{1}{16(1+\alpha)^2} \frac{1}{x^2} = 0$$

$$\alpha = -1 \quad \lim_{x\to+\infty} \frac{\frac{1}{4x^4}+o\left(\frac{1}{x^4}\right)}{\left(-\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)\right)^2} = \lim_{x\to+\infty} \frac{\frac{1}{4x^4}+o\left(\frac{1}{x^4}\right)}{\frac{4}{x^4}+o\left(\frac{1}{x^4}\right)} = \frac{1}{16}$$

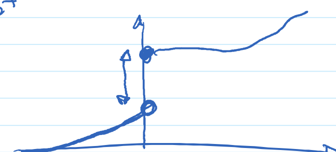
$f(x)$ is continuous in x_0 if $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = \ell < +\infty$ and $\ell = f(x_0)$

$\Uparrow$ D

$f(x)$ is derivable in x_0 if 1) $\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = \ell < +\infty$, $\ell = f'(x_0)$

2) $\lim_{x\to x_0^-} f'(x) = \lim_{x\to x_0^+} f'(x) = f'(x_0)$

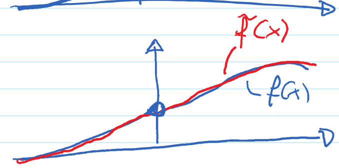
DISCONTINUITY: I $\quad \lim_{x\to x_0^-} f(x) = \ell_1 \neq \ell_2 = \lim_{x\to x_0^+} f(x)$



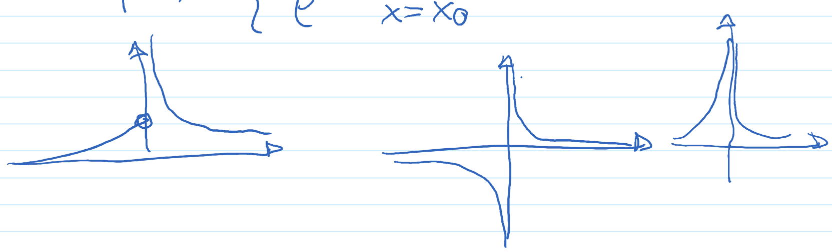
II $\quad \lim_{x\to x_0^-} f(x) = \ell = \lim_{x\to x_0^+} f(x)$

$\nexists f(x_0)$

$$\Longrightarrow \tilde{f}(x) = \begin{cases} f(x) & x \neq x_0 \\ \ell & x = x_0 \end{cases}$$



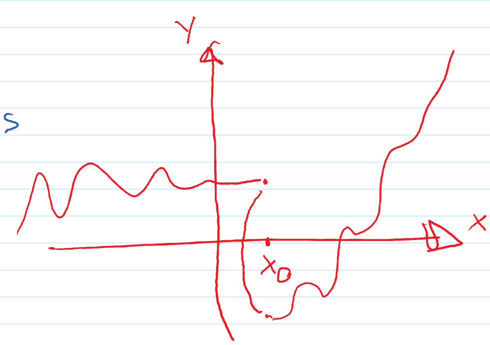
III



EXERCISE

$$f(x) = \begin{cases} (\alpha+1)\frac{x-1}{x^2+1} + \beta\sin(\pi x) & x \leq 1 \\ \alpha x + (\beta+1)\cos(\pi x) & x > 1 \end{cases}$$

find $a, b \in \mathbb{R}$ | $f(x)$ is continuous and derivable



$$\begin{cases} \lim_{x\to1^-} (\alpha+1)\frac{x-1}{x^2+1} + \beta\sin(\pi x) = (\alpha+1)\cdot 0 + \beta\cdot 0 = 0 \\ = \\ \lim_{x\to1^+} \alpha x + (\beta+1)\cos(\pi x) = \alpha\cdot 1 + (\beta+1)(-1) = \alpha - \beta - 1 \end{cases}$$

$\longrightarrow \alpha - \beta - 1 = 0$

$$\begin{cases}\displaystyle\lim_{x\to 1^-} (\alpha+1)\,\frac{1\cdot(x^2+1)-(x-1)2x}{(x^2+1)^2}+\beta\pi\cos(\pi x) \\ \displaystyle\lim_{x\to 1^+} \alpha+\underbrace{(\beta+1)(-\pi)\sin\pi x} = \alpha\end{cases}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$$

$$\frac{d}{dx}\cos(f(x))=\sin(f(x))\cdot f'(x)$$

$$= (\alpha+1)\,\frac{x^2+1-2x^2+2x}{(x^2+1)^2}+\beta\pi\cos(\pi x) = (\alpha+1)\,\frac{\cancel{-1}+2\cancel{+1}}{(1+1)^2}+\beta\pi(-1)$$

$x^2+1-2x^2+2x \to -x^2+2x+1$

$\frac{\cancel{-1}+2\cancel{+1}}{(1+1)^2} = \frac{1}{2}$

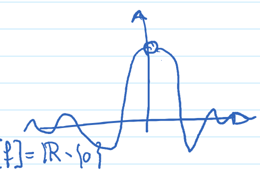
$$= \frac{1}{2}\alpha+\frac{1}{2}-\beta\pi$$

$$\longrightarrow \alpha=\frac{1}{2}\alpha+\frac{1}{2}-\beta\pi \longrightarrow \frac{1}{2}\alpha+\beta\pi-\frac{1}{2}=0$$

$$\Rightarrow\begin{cases}\alpha=\beta+1\\ \frac{1}{2}\alpha+\beta\pi-\frac{1}{2}=0\end{cases} \qquad \left(\frac{1}{2}+\pi\right)\beta+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}=0 \qquad \boxed{\begin{matrix}\alpha=1\\ \beta=0\end{matrix}}$$

NOTES:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$



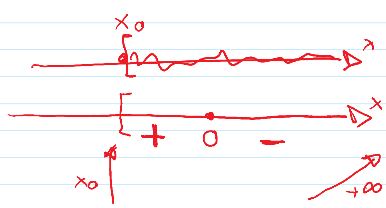
$\frac{\sin 0}{0} \nexists \qquad D[f]=\mathbb{R}\setminus\{0\}$

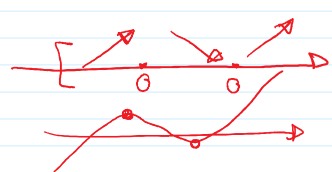
$$\tilde{f}(x)=\begin{cases}\frac{\sin x}{x} & x\neq 0\\ 1 & x=0\end{cases} \qquad D[\tilde{f}]=\mathbb{R}$$

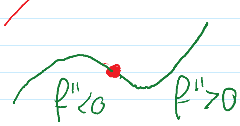
STUDY OF A FUNCTION

$f(x)$

1) $D[f]$
2) $sgn(f)$
3) $\lim(f)$
4) asyntotes
5) continuity
6) derivability
7) f'
8) monotonicity $\equiv sgn(f')$
9) extremes [min, max]
10) f''
11) convex / concave
12) flex points







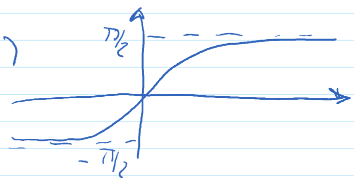
121 flex points

EXERCISE:

$$f(x) = \operatorname{arctg}\left(\frac{e^x}{e^x-1}\right)$$

1) $D[f]$ $\quad e^x-1\neq 0 \quad e^x\neq 1 \quad x\neq 0 \quad \Rightarrow \quad D[f]=\mathbb{R}\setminus\{0\}$

2) $\operatorname{sgn}(f)$



$\operatorname{arctg} y \geq 0 \iff y\geq 0$

$$\frac{e^x}{e^x-1}\geq 0 \qquad \begin{cases} N\geq 0 & \forall x \\ D\geq 0 & e^x\geq 1 \quad x\geq 0 \end{cases}$$

		0	
N	+		+
D	−	∄	+
	−	∄	+

3) $\lim(f)$

$$\lim_{x\to+\infty} \operatorname{arctg}\Big(\underbrace{\frac{e^x}{e^x-1}}_{\to 1}\Big) = \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$\lim_{x\to-\infty} \operatorname{arctg}\Big(\underbrace{\frac{e^x}{e^x-1}}_{\to 0}\Big) = 0$$

$$\lim_{x\to 0^+} \operatorname{arctg}\left(\frac{e^x}{e^x-1}\right) = \operatorname{arctg}(+\infty) = +\frac{\pi}{2}$$

↳ $e^x>1 \ \forall x>0 \Rightarrow e^x-1=0^+$ positive

$$\lim_{x\to 0^-} \operatorname{arctg}\left(\frac{e^x}{e^x-1}\right) = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

↳ $e^x<1 \ \forall x<0 \Rightarrow e^x-1=0^-$ negative

4) horizontal asymptote for $x\to+\infty$: $c=\frac{\pi}{4}$
for $x\to-\infty$: $c=0$

5) $f(x)$ is continuous but we cannot expand with continuity in $x=0$
↳ discontinuity of first order for $x=0$

6) $f(x)$ is combination of derivable functions, thus it is derivable in the domain

7) $$f'(x) = \frac{1}{1+\left(\frac{e^x}{e^x-1}\right)^2}\cdot\frac{e^x(e^x-1)-e^{2x}}{(e^x-1)^2} = \frac{\cancel{(e^x-1)^2}}{(e^x-1)^2+(e^x)^2}\cdot\frac{-e^x}{\cancel{(e^x-1)^2}}$$

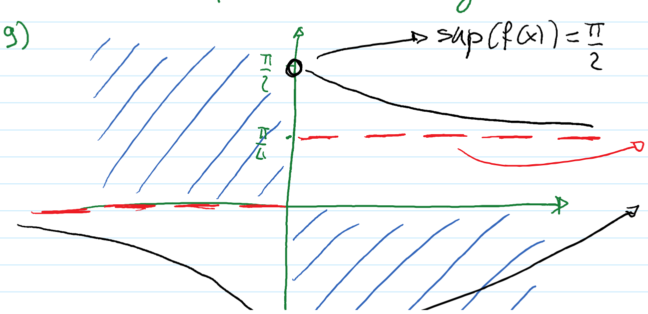
$$= \frac{-e^x}{(e^x-1)^2+(e^x)^2}$$

$N<0 \ \forall x$ strictly because $e^x>0$

$D\geq 0 \ \forall x$ (sum of squares)

8) $\Rightarrow f(x)$ is decreasing in all the domain

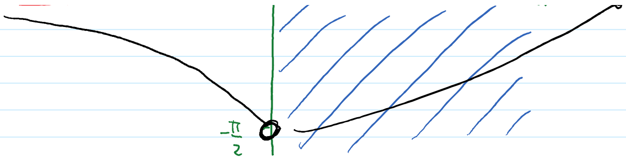
9)



$\sup(f(x)) = \frac{\pi}{2}$ not a maximum since $f(0)$ ∄

$f(x)$ cannot go below the asymptote because it is always decreasing!

$\inf(f(x)) = -\frac{\pi}{2}$



HOMEWORKS :

$$f(x)=\begin{cases}\dfrac{1}{x^2-x+a} & x\geq 1/2\\ 4\sin\pi x & x\leq 1/2\end{cases}$$

find $a\in\mathbb{R}$ | $f(x)$ is continuous

how is the graph of the function?

FIND IF $f(x)$ IS EXTENDABLE WITH CONTINUITY AT X_0

1) $f(x)=\begin{cases}\sin x & -1\leq x<0\\ \dfrac{(\sin x^2)^3}{(\operatorname{tg} x^3)^2} & 0\leq x\leq 1\end{cases}$, $X_0=0$

2) $f(x)=\dfrac{x\log x}{x^2-1}$, $X_0=1$

FIND PARAMETER | $f(x)$ IS CONTINUE AND DERIVABLE

1) $f(x)=\begin{cases}(\alpha-1)\operatorname{arctg} x+(\beta+1)x^2 & x\leq 1\\ \alpha x+\beta\sin(\pi x) & x>1\end{cases}$

2) $f(x)=\begin{cases}\alpha x+\sin x & x<0\\ x^2+\beta(e^x-1) & x>0\end{cases}$ $\vee$ $f(0)=0$

3) $f(x)=\begin{cases}\alpha x+(\alpha-1)x^2\sin\frac{1}{x} & x<0\\ x^2+\beta(e^x-1) & x>0\end{cases}$ $\vee$ $f(0)=0$

STUDY THE FUNCTIONS :

6) $f(x)=\arcsin\left(\dfrac{2x}{x^2+x+4}\right)$

7) $f(x)=x(2\log x-\log^3 x)$

8) $f(x)=\log\left(\dfrac{1}{\cos^4 x}\right)-\operatorname{tg}^2 x$

9) $f(x)=\operatorname{arctg}(2x+1)$

10) $f(x)=e^{-x}\log(1+2x)+e^x$