$$\lim \frac{\log(1+y)}{y} = 1$$

Derivatives

- $\exists$ derivative $f'(x_0) \Leftrightarrow f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$
- $(f+g)'(x) = f'(x) + g'(x)$, $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

- $(x^\alpha)' = \alpha x^{\alpha-1}$ $\quad (e^x)' = e^x$
- $(\log_e x)' = \frac{1}{x}$ $\quad (\log x)' = \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h}$

$$= \lim_{h \to 0} \frac{\log\left(x\left(1+\frac{h}{x}\right)\right) - \log x}{h} = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h/x} \cdot \frac{1}{x} \overset{y = \frac{h}{x}}{=} \lim_{y \to 0} \frac{\log(1+y)}{y} \cdot \frac{1}{x} = \frac{1}{x}$$

- $(\sin)'(x) = \cos x$ $\quad (\cos)'(x) = -\sin x$ $\quad (tg)'(x) = \frac{1}{\cos^2 x}$
- $\text{cotg}'(x) = -\frac{1}{\sin^2 x}$ $\quad (\sinh)' = \cosh(x)$ $\quad (\cosh)'(x) = \sinh(x)$

$$\left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

Theorem: $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ $(f(D) \subseteq E)$

$x_0 \in \operatorname{Int}(D)$, $y_0 = f(x_0) \in \operatorname{Int}(E)$

$$\boxed{\exists f'(x_0),\ \exists g'(y_0)} \Rightarrow \exists (g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

hypothesis — thesis

$$D \xrightarrow{\ f\ } E \xrightarrow{\ g\ } \mathbb{R}$$

$$x \longmapsto f(x) \longmapsto g(f(x)) = g \circ f(x)$$

$\cos\sqrt{x^2+3}$ $\quad g(y) = \cos y \quad f(x) = \sqrt{x^2+3}$

$= g \circ f(x)$ $\quad \tilde{g}(y) = \cos\sqrt{y} \quad \hat{f}(x) = x^2+3$

$= \tilde{g} \circ \hat{f}$

$g(f(x_0)) + g'(f(x_0))(f(x) - f(x_0)) + o(f(x) - f(x_0))$

Proof:

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{g(f(x_0)) + g'(f(x_0))(f(x) - f(x_0)) + o(f(x) - f(x_0)) - g(f(x_0))}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{g'(f(x_0)) \cdot \left(f'(x_0)(x - x_0) + o(x - x_0)\right) + o\left(f'(x_0)(x - x_0) + o(x - x_0)\right)}{x - x_0}$$

$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$

$g(y) = g(y_0) + g'(y_0)(y - y_0) + o(y - y_0)$

$$= g'(f(x_0))\, f'(x_0) \frac{(x-x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g'(f(x_0))\cdot o(x-x_0)}{x-x_0} \to 0 \; +$$

$$\lim_{x\to x_0} \frac{o\left(f'(x_0)(x-x_0) + o(x-x_0)\right)}{x-x_0} =$$

$$= 0$$

Observe there

$$\frac{o\left(f'(x_0)(x-x_0)+o(x-x_0)\right)}{f'(x_0)(x-x_0)+o(x-x_0)} \cdot \frac{f'(x_0)(x-x_0)+o(x-x_0)}{(x-x_0)}$$

$$\to 0 \qquad \to f'(x_0)$$

$$= g'(f(x_0)) \cdot f'(x_0). \qquad \text{q.e.d}$$

$$\frac{d\left(e^{\cos(x^3)}\right)}{dx} = \frac{dg}{dy}(f(x)) \cdot \frac{df}{dx}(x) = (*)$$

$$e^{\cos(x^3)} = g(f(x))$$

with $e^y = g(y)$

$\cos x^3 = f(x)$

$$\frac{dg}{dy}(y) = e^y \qquad \frac{df}{dx} = (\cos x^3)' = (h \circ k)'(x) = h'(k(x)) \cdot k'(x)$$

$$= -\sin(x^3) \cdot 3 \cdot x^2$$

$h(z) = \cos z \quad k(x) = x^3$

(*) $= -3e^{\cos(x^3)} \cdot \sin(x^3)\, x^2$

Observe

$$(k \circ g \circ f)'(x) = k'(g(f(x))) \cdot g'(f(x)) \cdot f'(x).$$

Teorema: $f: D \to \mathbb{R}$, $x_0 \in \text{int}(D)$

f differentiable at $x_0 \Longrightarrow f$ continuous at x_0

Proof.

$$\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\to f'(x_0)} \underbrace{(x-x_0)}_{\to 0} =$$

$$= f'(x_0)\cdot 0 = 0$$

q.e.d

Alternatively

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} f(x_0) + \underbrace{f'(x_0)(x-x_0)}_{\to 0} + \underbrace{o(x-x_0)}_{\to 0} =$$

$$= f(x_0)$$

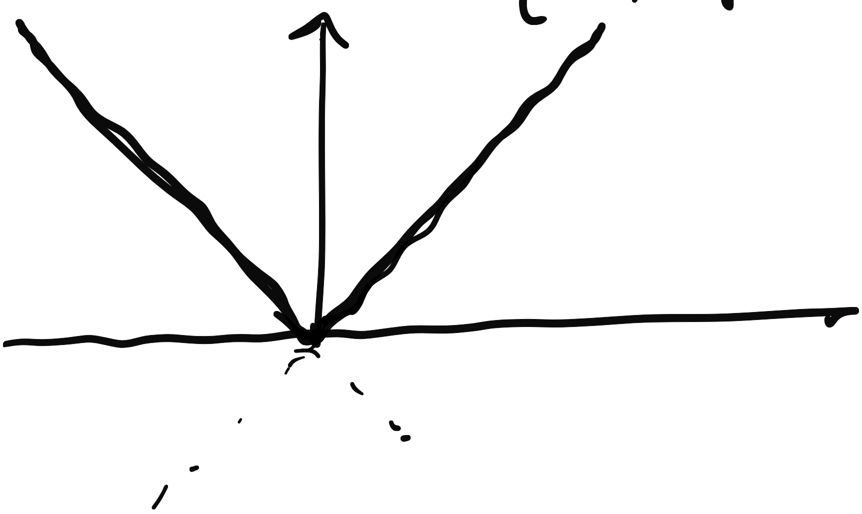
q.e.d

---

f continuous at $x_0 \overset{?}{\Longrightarrow} f$ differentiable at x_0

No:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



in $x = 0$ it is not differ...

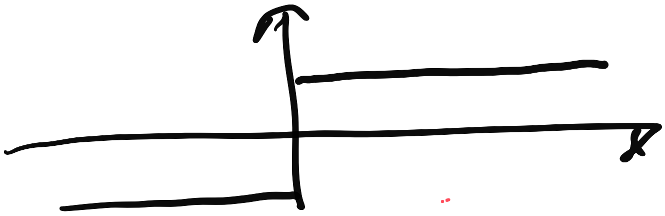
$$\lim_{x\to 0} \frac{|x|}{x}$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim \frac{x}{x} = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim \frac{-x}{x} = -1$$

$x \longmapsto |x|$

$$(|x|)' = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$



$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Fermat, Rolle, Lagrange, Cauchy.

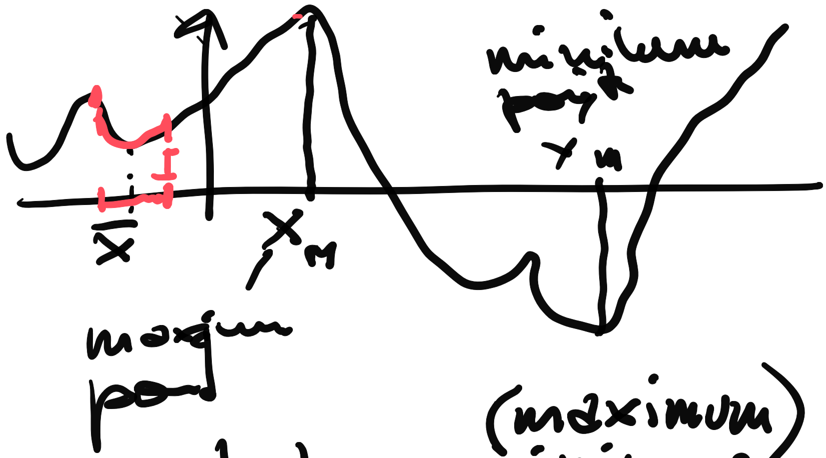
Lagrangia

Fermat Theorem

$f: D \longrightarrow \mathbb{R}$, $x_0 \in D$ is

a (absolute) minimum point if $f(x_0) \leq f(x)$ $\forall x \in D$

" (absolute) maximum " " $f(x_0) \geq f(x)$ $\forall x \in D$



a relative minimum (maximum) point $\bar{x}$ if $\exists I$ neigh. of $\bar{x}$, $f(\bar{x}) \le f(x)$ $\forall x \in I \cap D$

$(f(\bar{x}) \ge f(x)$ " " ")

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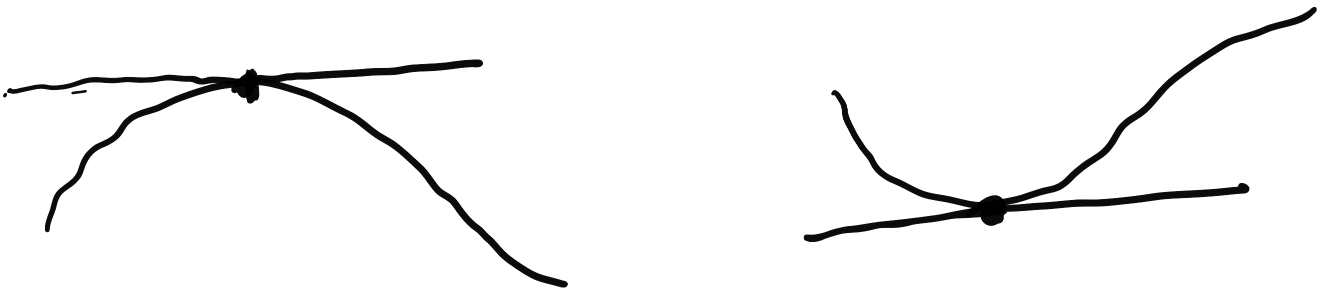
relative = local

$\bar{x}$ absolute maximum $\Rightarrow$ $\bar{x}$ relative maximum

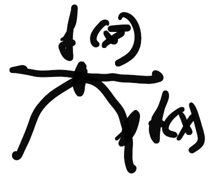
" " minimum $\Rightarrow$ $\bar{x}$ relative minimum



Theorem (Fermat). $f : D \to \mathbb{R}$ and let $\bar{x} \in Int(D)$ and assume that f is differentiable at $\bar{x}$ and $\bar{x}$ is a relative maximum or minimum

Then $f'(\bar{x}) = 0$

Proof (case of rel. max)



$$f'(\bar{x}) = \lim_{x \to \bar{x}+} \frac{\overbrace{f(x) - f(\bar{x})}^{\leq 0}}{\underbrace{x - \bar{x}}_{>0}} \Rightarrow f'(\bar{x}) \leq 0$$

$$f'(\bar{x}) = \lim_{x \to \bar{x}-} \frac{\overbrace{f(x) - f(\bar{x})}^{\leq 0}}{\underbrace{x - \bar{x}}_{<0}} \Rightarrow f'(\bar{x}) \geq 0$$

$$\Rightarrow f'(\bar{x}) = 0$$

$$f(x) = x^3 + x^2$$

$$f'(x) = 3x^2 + 2x = 0$$

$$\Leftrightarrow \quad x = 0 \qquad x = -\frac{2}{3}$$

Theorem Rolle

$f : [a,b] \to \mathbb{R}$, f continuous on $[a,b]$,

f differentiable on $]a,b[$.

$f(a) = f(b)$

Then there exists a point $\bar{x} \in]a,b[$ s.t.

$$f'(\bar{x}) = 0$$

Proof: By Weierstrass Th. there exist a maximum point $x_M \in [a,b]$ and a minim point $x_m \in [a,b]$ of f.

I) $\max = f(x_M) = f(x_m) = \min$

$\Longrightarrow f = k \in \mathbb{R}$ $\quad f'(x) = 0 \ \forall x \in]a,b[$

II) $\min = f(x_m) < f(x_M)$

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