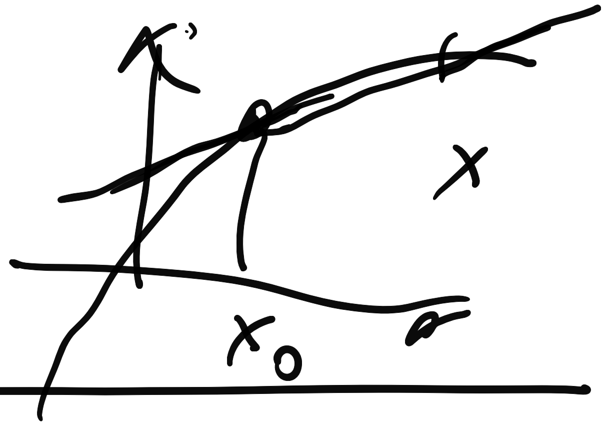
$f: D \longrightarrow \mathbb{R}$ $\quad x_0 \in D$

$$\lim_{x \to x_0} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\text{differential quotient}} = f'(x_0) = \frac{df}{dx}(x_0)$$

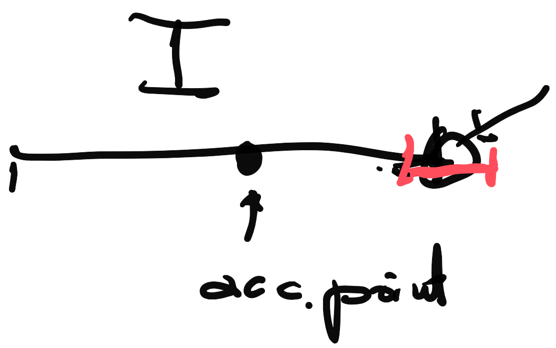


$E \subseteq \mathbb{R}$

- $x_0 \in E$
- x_0 is accumulation point for E $\left(\begin{array}{l} \text{it maybe } x_0 \in E \\ \text{" "} \quad x_0 \notin E \end{array}\right)$

$\forall I$ neigh.. of x_0

$$E \cap (I \setminus \{x_0\}) \neq \emptyset$$



not included but acc. point

introduced to define LIMITS

- we check $\boxed{\text{continuity}}$ of $f: D \to \mathbb{R}$ at $x_0 \in acc D \cap D$

$$D = [-3,-1] \cup \{0\} \cup [1,2[$$

$0 \in D$ $\qquad 0 \notin acc(D)$.

- if we speak of differentiability

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

x_0 MUST <u>BELONG</u> TO D TOGETHER WITH A NEIBOORHOOD

I of x_0.

Such points x_0 are called interior points

We write

$$x_0 \in \operatorname{Int}(D)$$

$f(x) = \frac{1}{x}$ $\qquad$ $f : \underbrace{\mathbb{R}\setminus\{0\}}_{D} \longrightarrow \mathbb{R}$

$\operatorname{int} D =$

$r = \frac{|x_0|}{2}$ $\qquad$ $[x_0 - r, x_0 + r] = I$

$$I \subseteq \mathbb{R}\setminus\{0\}$$

$\text{int}\, D = D = \mathbb{R} \setminus \{0\}$.

$x_0 \neq 0 \quad \lim_{x \to x_0} \dfrac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} =$

$= \lim_{x \to x_0} \dfrac{\frac{x_0 - x}{x x_0}}{x - x_0} = -1 \lim_{x \to x_0} \dfrac{1}{x x_0} =$

$= -\dfrac{1}{(x_0)^2}$

$\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$

$\left(x^{-1}\right)' = -x^{-2}$

Claim:

$(*) \quad \left(x^{-n}\right)' = \left(\dfrac{1}{x^n}\right)' = -n x^{-n-1}$

Prove the claim:

Let us try induction

Step 1 $n=1$ o.k. (see above)

Step 2 Suppose ~~Suppose~~ $(*)$ for n and prove for $n+1$

we want to prove

$$\boxed{\left(x^{-(n+1)}\right)' = -(n+1)\,x^{-(n+2)}}$$

$$\left(x^{-(n+1)}\right)' = \Big(\underbrace{x^{-n}}_{f}\cdot\underbrace{x^{-1}}_{g}\Big)' =$$

$$-n\,x^{-n-1}\,x^{-1} + x^{-n}\left(-1x^{-2}\right) =$$

$$= -n\,x^{-(n+2)} + \left(-\,x^{-(n+2)}\right) =$$

$$= -(n+1)\, x^{-(n+2)}$$

q.e.d.

$\alpha \in \mathbb{R} \qquad f(x) = x^{\alpha}$

$f : [0, +\infty[\longrightarrow \mathbb{R}$

$x_0 > 0 \qquad f'(x_0) = \alpha\, x^{\alpha - 1}$

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{(x_0+h)^{\alpha} - x_0^{\alpha}}{h} =$$

$$= \lim_{h \to 0} x_0^{\alpha} \frac{\left(1 + \frac{h}{x_0}\right)^{\alpha} - 1}{h} =$$

$$\lim_{x \to 0} x_0^{\alpha} \frac{\cancel{1} + \alpha \frac{h}{x_0} + o\left(\frac{h}{x_0}\right) - \cancel{1}}{h} =$$

$$= x_0^{\alpha} \left(\frac{\alpha}{x_0} + \lim_{x \to x_0} \frac{o(h)}{h} \right)$$

(observe that $o\left(\frac{h}{x_0}\right) = o(h)$)

$$= \frac{x_0^{\alpha}\,\alpha}{x_0} = \alpha\, x_0^{\alpha-1}$$

$$(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$$

proved!

---

f and g differentiable at x_0

x_0 interior to the domain of f and g

$$\left(\frac{f}{g}\right)'(x_0)$$

$$\lim_{x \to x_0} \frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} =$$

($g \neq 0$ in a neighbourhood of x_0)

$$= \lim_{x \to x_0} \frac{\dfrac{f(x)g(x_0) - f(x_0)\cdot g(x)}{g(x)g(x_0)}}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{\dfrac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{1}{g(x)g(x_0)} \left(\frac{\big(f(x) - f(x_0)\big)g(x_0)}{x - x_0} - \frac{g(x) - g(x_0)}{x - x_0} \cdot \boxed{f(x_0)} \right) =$$

(g is continuous at x_0 because)

$$= \boxed{\frac{1}{(g(x_0))^2} \left(f'(x_0)g(x_0) - g'(x_0)f(x_0) \right)}$$

$$\left(\frac{\cancel{f}}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{\left(g(x)\right)^2}$$

---

if $f = 1$

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{\left(g(x)\right)^2} \quad (**)$$

$$\left(x^{-n}\right)' = \left(\frac{1}{x^n}\right)' \qquad n \in \mathbb{N}$$

$$\overset{(**)}{=} -\frac{n x^{n-1}}{x^{2n}} = -n x^{-n-1}.$$

this is another method for proving the above formula (*)

Proposition: $f : D \to \mathbb{R}$

$x_0 \in \operatorname{int}(D)$

$$\boxed{f \text{ is differentiable at } x_0} \iff \boxed{f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)}$$

Proof: " $\Rightarrow$ "

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\Updownarrow$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} f'(x_0) \frac{(x-x_0)}{x-x_0}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0)}{x - x_0} = 0$$

i.e. $f(x) - f(x_0) - f'(x_0)(x-x_0) = o(x-x_0)$

$$\Updownarrow$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

" $\Leftarrow$ "

We know
$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f'(x_0)(x-x_0)+o(x-x_0)}{x-x_0}$$

$$= f'(x_0) + \lim_{x\to x_0} \underbrace{\frac{o(x-x_0)}{x-x_0}}_{\to 0} =$$

$$= f'(x_0)$$

Recall Taylor expansion:

$\sin x = x + o(x)$ Let's prove it

$f(x) = \sin x \quad x_0 = 0$

$$\sin x = \sin 0 + 1 \cdot x + o(x) =$$

$$= x + o(x)$$

$x_0=0 \quad tg\, x = tg(0) + (tg)'(0) \cdot x + o(x)$

$$(tg\, x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x - (-\sin x)\sin x}{\cos^2 x} =$$

$$= \frac{1}{\cos^2 x}$$

$$\Rightarrow tg\, x = 0 + \frac{1}{\cos^2 0} x + o(x) =$$

$= x + o(x)$

<u>Problem</u> find the derivative of

$$f(x) = \sin x^2$$

$$\cancel{f'(x) = \cos x^2}$$

$$\sin x^2 = g \circ f(x)$$

where $f(x) = x^2$ $g(y) = \sin y$

$$e^{\sqrt{x^4 + |\sin x|}} = g \circ f$$

$$g(y) = e^y \qquad f(x) = \sqrt{x^4 + |\sin x|}$$

$$e^{\sqrt{x^4 + |\sin x|}} = h \circ k \circ l(x)$$

$$l(x) = x^4 + |\sin x|$$

$$k(y) = \sqrt{y} \quad , \quad h(z) = e^z$$

Theorem: $f: D \to \mathbb{R}$ $g: E \to \mathbb{R}$

$f(D) \subseteq E$

$g \circ f: D \to \mathbb{R}$

$g \circ f(x) = g(f(x))$

$x_0 \in \mathrm{int}(D)$

$f(x_0) = y_0 \in \mathrm{int}(E)$

If f is diff. at x_0 and g is diff. at y_0, then $g \circ f$ is diff. at x_0 and

$$(g \circ f)' = \underline{g'(f(x_0))} \cdot \underline{f'(x_0)}$$

$$\boxed{\left(\sin x_0^2\right)'} = (g \circ f)'(x_0) =$$

$g(y) = \underline{\sin y}$

$f(x) = x^2$

$$= \boxed{\cos(x_0^2) \cdot 2x_0}$$