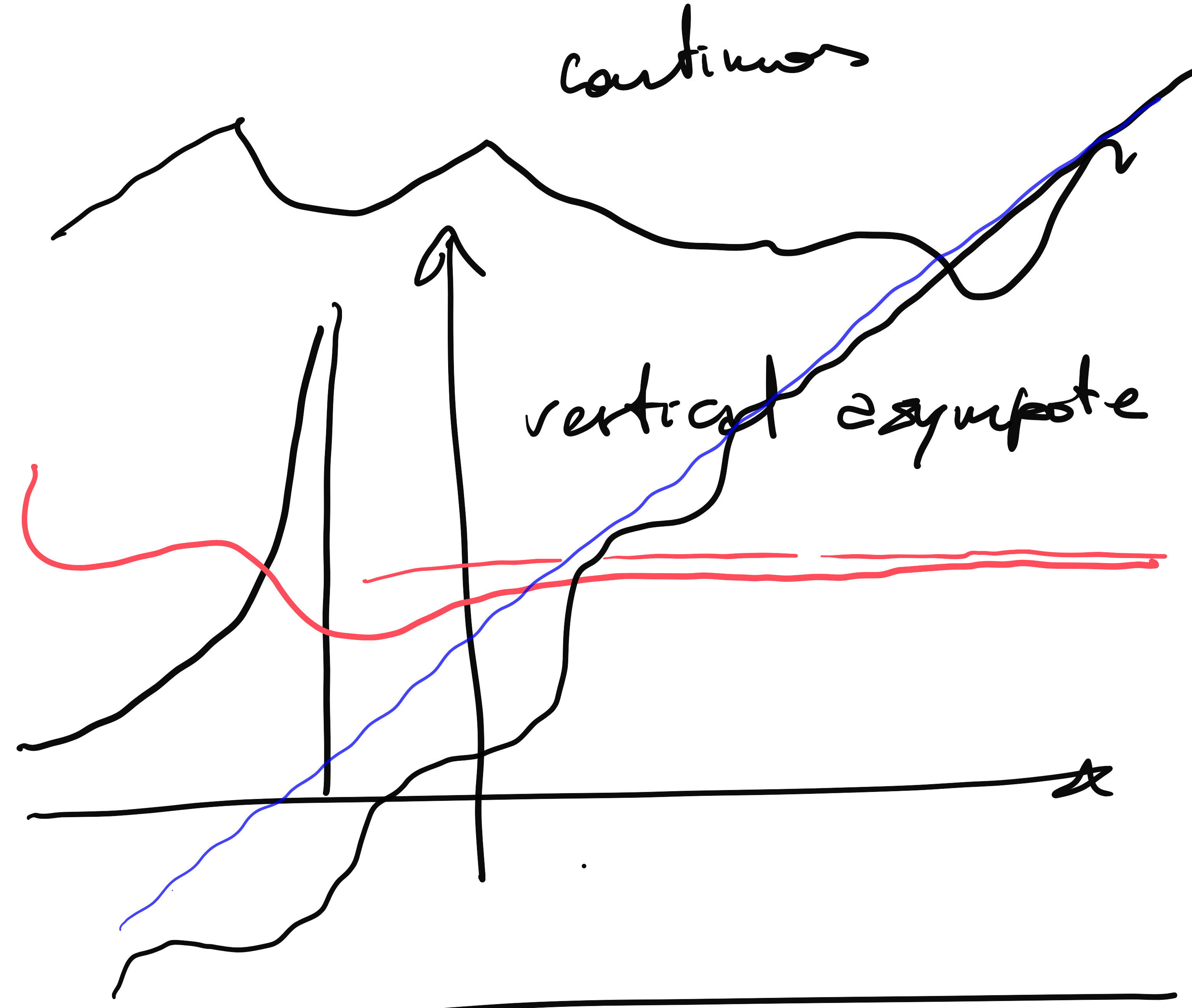
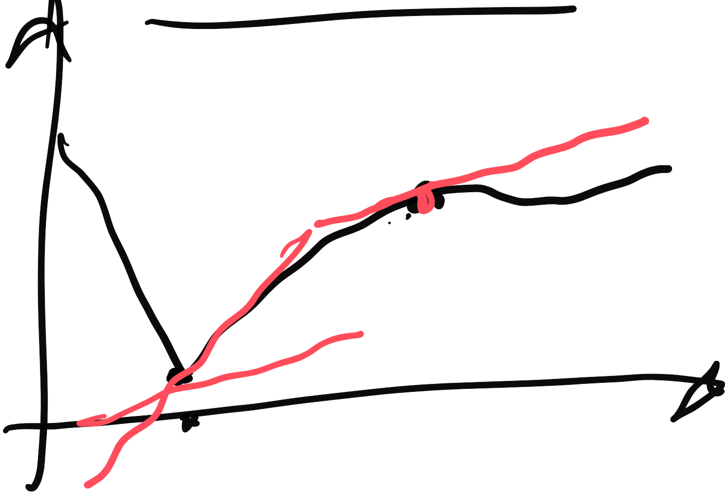
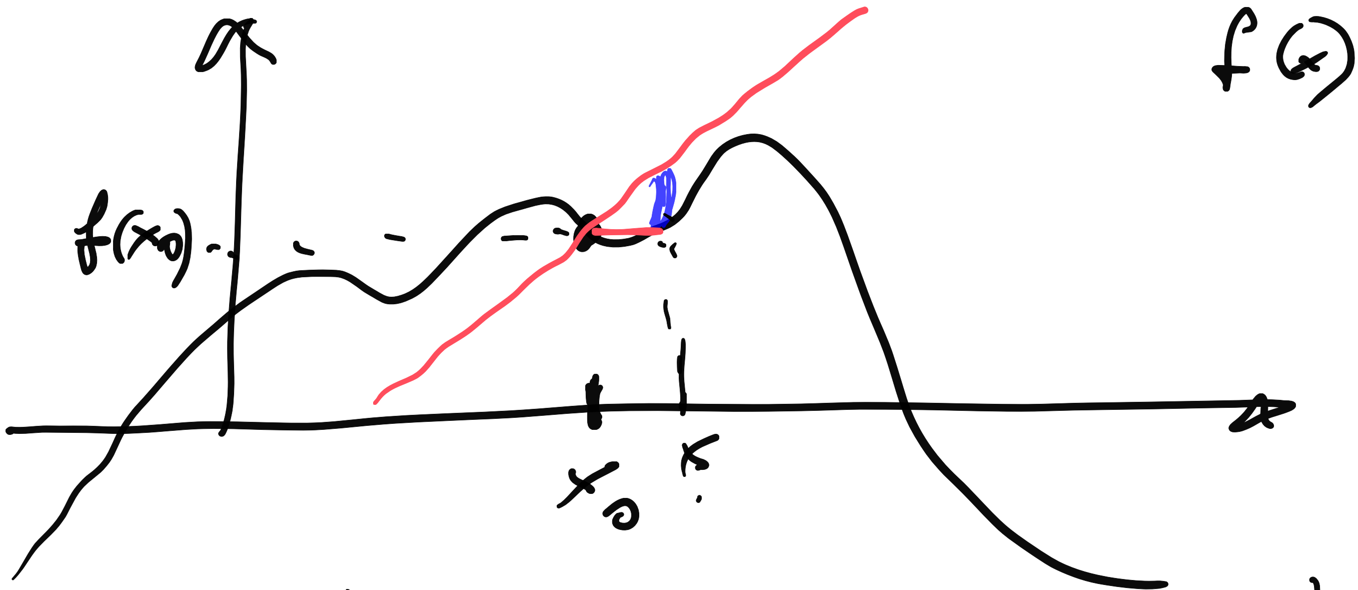


Newton

Leibniz





$(x_0, f(x_0))$

$$y = \underset{\text{angular coefficient}}{m}(x - x_0) + f(x_0)$$

$$\varepsilon_m(x) = f(x) - \big(m(x-x_0) + f(x_0)\big)$$

$$\lim_{x \to x_0} \varepsilon_m(x) = 0$$

If $\exists$ find $\hat{m}$ s.t

$$\lim_{x \to x_0} \frac{\varepsilon_{\hat{m}}(x)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{f(x) - \hat{m}(x - x_0) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - \hat{m} \right|$$

$= 0$ if and only if

$$\hat{m} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

We call

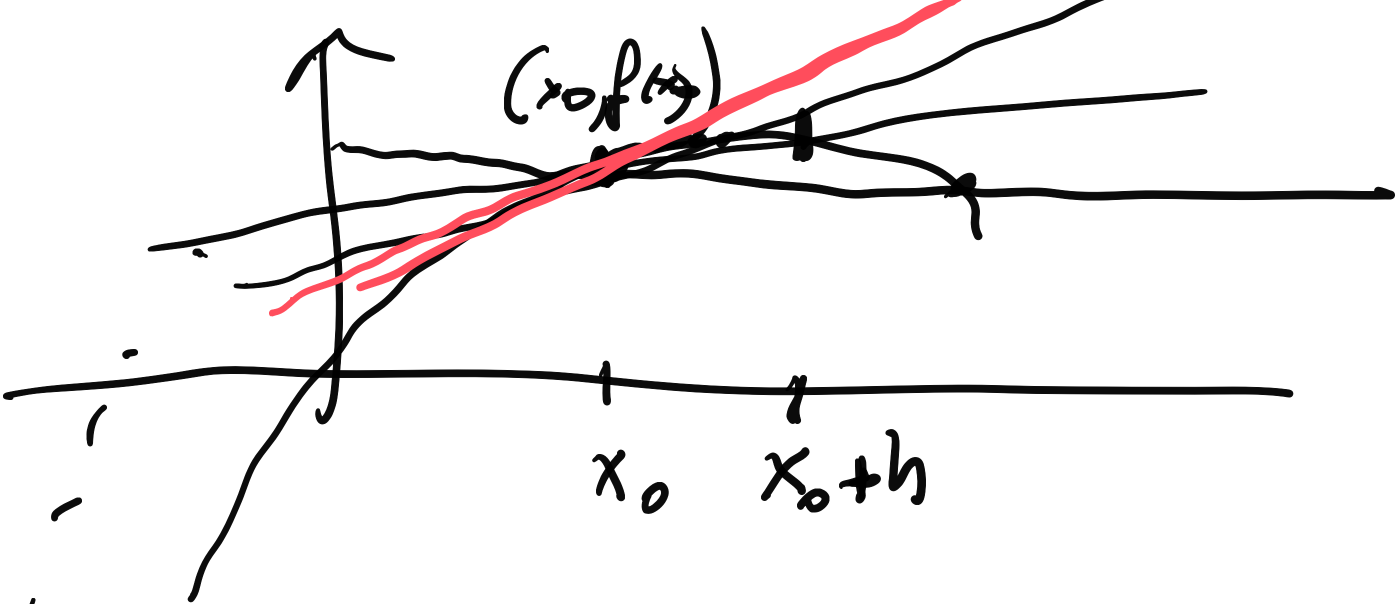
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$ = the **DERIVATIVE** of f at x_0

f is said **DIFFERENTIABLE**

Notation:

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\frac{df}{dx}(x_0) := \text{''}$$



has angular coeff.:

$$y_{(h)} = \left(\frac{f(x_0+h) - f(x_0)}{\cancel{x_0} + h - \cancel{x_0}}\right)(x - x_0) + f(x_0)$$

$h \to 0$

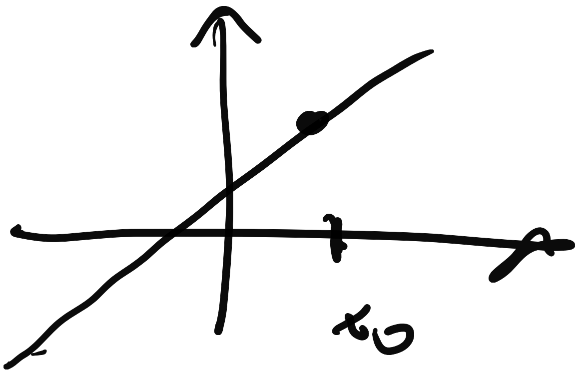
$$\boxed{y = f'(x_0)\,(x - x_0) + f(x_0)}$$

equation of the tangent at $(x_0, f(x_0))$

$$\left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \overset{x = x_0 + h}{=} \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}\right)$$

Compute derivatives

$$f(x) = \underline{\underline{2}}x + b$$



$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =$$

$$\lim_{x \to x_0} \frac{2x \cancel{+b} - 2x_0 \cancel{-b}}{x - x_0} = 2$$

$$f(x) = x^2$$

$$\boxed{f'(x_0)} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} =$$

$$= \lim_{x \to x_0} x + x_0 = \boxed{2x_0}$$

$$(x^2)' = 2x$$

$$f(x) = x^3$$

$$f'(x_0) = \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} =$$

$$= \lim_{x \to x_0} x^2 + x_0 x + x_0^2 = 3x_0^2$$

Up to now $(x^n)' = n x^{n-1}$

is it a general rule?

$f(x) = x^n$

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \frac{(x_0+h)^n - x_0^n}{h} =$$

$$= \lim_{h\to 0} \frac{1}{h} \left(\sum_{k=0}^{n} \binom{n}{k} x_0^{n-k} \cdot h^k - x_0^n \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} h^k =$$

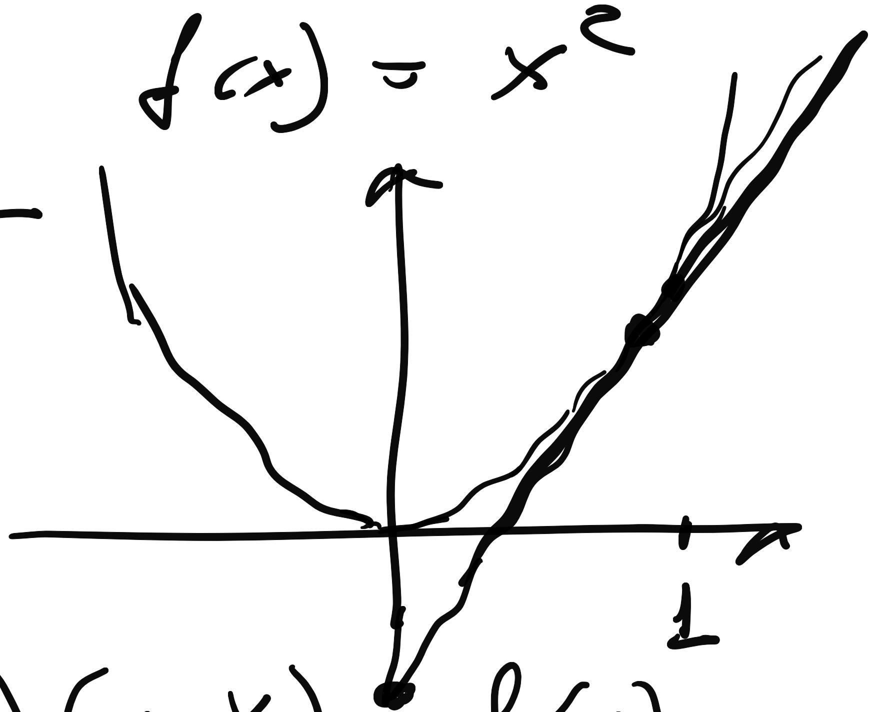
$$= \lim_{h\to 0} \left(\frac{1}{\cancel{h}} n x_0^{n-1} \cancel{h} + \underbrace{\frac{1}{h} \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} h^k}_{\to 0} \right)$$

$$= \boxed{n x_0^{n-1}}$$

We have proved that

$$(x^n)' = n x^{n-1}$$

Exercise: Find the tangent line to $f(x) = x^2$ at $x_0 = 1$



$$y = f'(x_0)(x - x_0) + f(x_0)$$

$$y = 2x_0 (x - x_0) + f(x_0)$$

$x_0 = 1$

$$y = 2(x-1) + 1 =$$

$$= 2x - 1$$

$$y = 2x - 1$$

Equation of the tang. at $x_0 = 1$

Exercise: Find the tangent to $f(x) = x^3$ at $x_0 = 5$

$$f'(x) = 3x^2$$

$$y = 75x - 250$$

$$y = f'(5)(x-5) + f(5) =$$

$$= 3 \cdot 5^2 (x-5) + 125 =$$

$$= 75x - 375 + 125 =$$

$$= 75x - 250$$

$$f(x) = k \in \mathbb{R}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = 0$$

$$\left(k x^n\right)' = k\left(x^n\right)'$$

More generally

$$\left(k \cdot f(x)\right)' = k\, f'(x)$$

$$= \lim_{h\to 0} \frac{k f(x+h) - k f(x)}{h} =$$

$$k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = k \cdot f'(x)$$

If if is true

Theorem 1

f and g differentiable at x_0, then the function

$$x \longmapsto f(x) + g(x) = (f+g)(x)$$

is differentiable

$$(f+g)'(x) = f'(x) + g'(x)$$

Proof:

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

Derivatives of polynomials

$$(a_0 + a_1 x + \cdots + a_n x^n)' \underset{\text{by Th. 1}}{=} (a_0)' + (a_1 x)' + \cdots + (a_n x^n)' =$$

$$0 + a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

$f(x) = \sin x$

$$(\sin x)' = \cos x$$

Indeed

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$= \lim_{h \to 0} \frac{\overbrace{\sin x \cos h}^{\sin x} + \sin h \cos x - \sin x}{h}$$

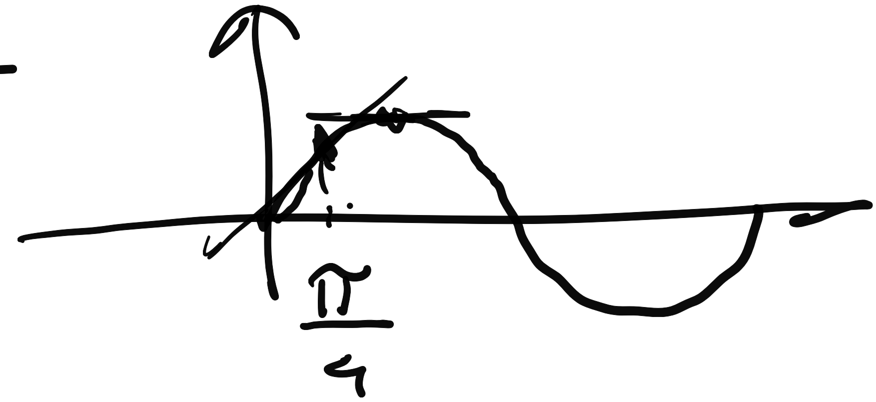
$$= \lim_{h \to 0} \sin x \frac{(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h}{h} \cos x$$

$$= \lim_{h \to 0} \sin x \underbrace{\underbrace{\frac{\cos h - 1}{h^2}}_{-\frac{1}{2}} h}_{\to 0} + \cos x =$$

$$= \cos x$$

Find the tang. to $\sin x$

at $x = \frac{\pi}{4}$



$$(\cos x)' = \frac{d(\cos)}{dx}(x) =$$

two notations of the same object

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} =$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} -\sin x \underbrace{\frac{\sin h}{h}}_{\to 1} + \cos x \underbrace{\left(\frac{\cos h - 1}{h}\right)}_{\to 0}$$

$$= -\sin x$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(e^x)' = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

$$= e^x \lim_{h \to 0} \underbrace{\frac{e^h - 1}{h}}_{\to 1} = e^x.$$

$$f(x) = 5e^x + 4\cos x$$

$$f'(x) = (5e^x + 4\cos x)' \underset{\text{Th } 1}{=}$$

$$(5e^x)' + (4\cos x)' = 5(e^x)' + 4(\cos x)'$$

$= 5e^x - 4\sin x$.

Theorem f and g differentiable at x_0. Then $x \longmapsto (fg)(x) := f(x) \cdot g(x)$ is differentiable and

$$(fg)'(x_0) = g(x_0) \cdot f'(x_0) + f(x_0) \cdot g'(x_0)$$

$$\left(x^2 \sin x\right)' = 2x \sin x + x^2 \cos x$$

Proof

$$(fg)'(x_0) := \lim_{h \to 0} \frac{(f \cdot g)(x_0+h) - f \cdot g(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0+h)\, g(x_0+h) + f(x_0+h)\, g(x_0) - f(x_0+h) g(x_0) - f(x_0) g(x_0)}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(x_0+h)\left(g(x_0+h) - g(x_0)\right)}{h} + \frac{\left(f(x_0+h) - f(x_0)\right) g(x_0)}{h} \right) =$$

$$\lim_{h \to 0} f(x_0+h) \cdot \frac{g(x_0+h) - g(x_0)}{h} + g(x_0) \frac{f(x_0+h) - f(x_0)}{h}$$

= Suppose we knowed that differentiable at $x_0 \Longrightarrow$ continuous at x_0

$$= f(x_0) g'(x_0) + g(x_0) f'(x_0)$$

q.e.d