
MATHEMATICAL PHYSICS

Control Systems Engineering

First partial exam - 11/11/2022

Exercise 1. Let

$$V_k(x) = \frac{k}{3}x^3 - x, \qquad x \in \mathbb{R} \text{ and } k \in \mathbb{R}.$$

(a) Draw the bifurcation diagram for $\dot{x} = V_k(x)$.

(b) Draw the cobweb plot for the discrete dynamical system given by the iteration of the map $V_1(x)$. Determine equilibria and their (linear) (un)stability.

(c) Draw the phase-portraits for $\ddot{x} = -V_k'(x)$ corresponding to $k > 0$, $k < 0$ and $k = 0$.

(d) Linearize $\ddot{x} = -V_1'(x)$ around $(\pm 1, 0)$. Establish the quality of these equilibria for the linearized system; say –if it makes sense– the winding direction.

(e) Establish for which values of $v \in \mathbb{R}$ the solution of $\ddot{x} = -V_1'(x)$ with initial datum $(1, v)$ is periodic.

(f) How many orbits of $\ddot{x} = -V_1'(x)$ correspond to the energy value $E = V_1(-1)$?

Exercise 2. Let consider the vector field on $\mathbb{R}^3$:

$$X(x,y,z) = \begin{pmatrix} yz^2 \\ -(x-1)^2xz^2 \\ (2-x)x^2yz \end{pmatrix}$$

(a) Prove that $F(x,y,z) = x^2 + y^2 - z^2$ is a first integral for X.

(b) Give the definition of (topological) stability and asymptotic stability. Explain why the equilibrium $(1,0,2)$ for X cannot be asymptotically stable.

(c) Can the dynamics corresponding to such X have a limit cycle?

Exercise 3. Give the definition of invariant set for a continuous flow $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$. Let $\beta \in \mathbb{R}$ be fixed. For the vector field on $\mathbb{R}^2$:

$$X(x,y) = \begin{pmatrix} -3x + y \\ (\beta - 3)y \end{pmatrix}$$

show that the line $y = \beta x$ is an invariant set.

Exercise 4. Sketch a phase portrait in the plane consistent with the following information: three equilibria, one saddle and two stable nodes.