Geometria 1 - mod. A - Lezione 17

Note Title

Risolvere il seguente sistema con due parametri a, b (a coeff in $\mathbb{R}$)

$$\begin{cases} (1-a)z + b = 3 \\ ax + ay - 1 = -b \\ x + by = 2-a \end{cases} \qquad \left(\begin{array}{ccc|c} 0 & 0 & 1-a & 3-b \\ a & a & 0 & +1-b \\ 1 & b & 0 & 2-a \end{array}\right)$$

I step: ridurre a scala con operaz.: scambio righe e (iii)

MAI LAVORARE con le colonne

$$\overset{\mathrm{II}\leftrightarrow\mathrm{I}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & b & 0 & 2-a \\ a & a & 0 & 1-b \\ 0 & 0 & 1-a & 3-b \end{array}\right) \overset{\mathrm{II}-a\mathrm{I}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & b & 0 & 2-a \\ 0 & a-ab & 0 & 1-b-2a+a^2 \\ 0 & 0 & 1-a & 3-b \end{array}\right) = \left(\begin{array}{ccc|c} 1 & b & 0 & 2-a \\ 0 & a(1-b) & 0 & 1-b+a(-2+a) \\ 0 & 0 & 1-a & 3-b \end{array}\right)$$

- Osservo che se $\boxed{a(1-b) \neq 0 \quad e \quad 1-a \neq 0}$

$\Rightarrow$ il sistema ammette unica soluzione

$$\begin{cases} x = -by + (2-a) \\ \cancel{a(1-b)}\, y = \dfrac{1-b+a(-2+a)}{a(1-b)} \;(*) \\ \cancel{(1-a)}\, z = \dfrac{3-b}{1-a} \end{cases}$$

$rg\, A = 3 = rg(\text{matrice completa})$ $\begin{pmatrix} * \\ * \\ * \end{pmatrix} + \underbrace{Sol(A|0)}_{= 0}$

parametri è $= \#\text{incognite} - rg\, A = 3 - 3 = 0$

La soluzione se $a \notin \{0,1\}$ e $b \neq 1$ è

$$\begin{pmatrix} 2-a-b(*) \\ \dfrac{1-b+a(-2+a)}{a(1-b)} \\ \dfrac{3-b}{1-a} \end{pmatrix}$$

Ad esempio se $a = 5$ e $b = 0$ $\begin{pmatrix} -3 \\ \frac{16}{5} \\ \frac{3}{-4} \end{pmatrix}$

- Ora studio cosa accade se $a = 0$ oppure $b = 1$ oppure $a = 1$

$$\left(\begin{array}{ccc|c} 1 & b & 0 & 2-a \\ 0 & a(1-b) & 0 & 1-b+a(-2+a) \\ 0 & 0 & 1-a & 3-b \end{array}\right)$$

Caso 1) $\boxed{b=1}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2-a \\ 0 & 0 & 0 & a(a-2) \\ 0 & 0 & 1-a & 2 \end{array}\right) \overset{\mathrm{II}\leftrightarrow\mathrm{III}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2-a \\ 0 & 0 & 1-a & 2 \\ 0 & 0 & 0 & a(a-2) \end{array}\right)$$

se $a \notin \{0,2\}$ non ho soluzioni ⟵ non nullo

$a=0$ ho infinite soluz $\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \begin{array}{c} rg\, A = 2 \\ \| \\ rg\, matr.\, compl. \end{array}$

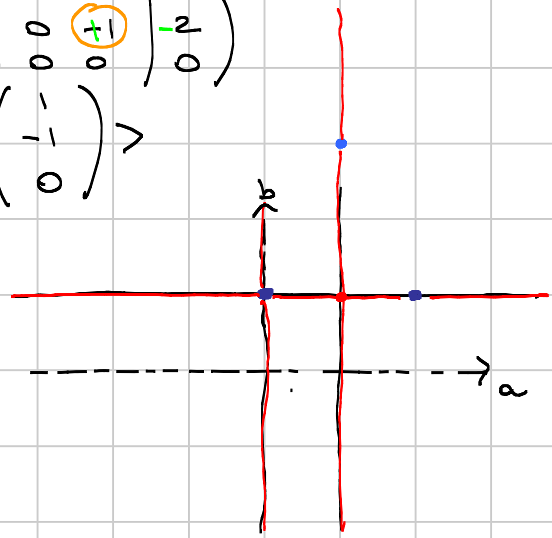
$\begin{array}{ccc} x & y & z \end{array}$

$$\begin{cases} x+y=2 \\ z=2 \end{cases} \qquad \begin{pmatrix} 2-\alpha \\ \alpha \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

$a=2$ ho infinite soluz $\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & +1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{pmatrix} -\alpha \\ \alpha \\ -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$



Per tutte le coppie esterne alle 3 rette il sistema ammette unica soluzione

- infinite soluzioni
- non ho soluzioni

Caso II $a=0$

$$\left(\begin{array}{ccc|c} 1 & b & 0 & 2-a \\ 0 & a(1-b) & 0 & 1-b+a(-2+a) \\ 0 & 0 & 1-a & 3-b \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & b & 0 & 2 \\ 0 & 0 & 0 & 1-b \\ 0 & 0 & 1 & 3-b \end{array}\right)$$

- $b \neq 1$ il sistema non ha soluzioni
- $b=1$ il sistema ha infinite soluz. già trovate

Caso III $a=1$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & b & 0 & 1 \\ 0 & 1-b & 0 & \cancel{1}-b-\cancel{1} \\ 0 & 0 & 0 & 3-b \end{array}\right)$$

- $b=1$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right)$ no soluzioni (già visto)
- $b=3$ $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad rg\, A = 2 = rg\, incompl.$

$$\begin{cases} x+3y=1 \\ +2y=+3 \\ 0=0 \end{cases} \qquad \left\{ \begin{pmatrix} 1-9/2 \\ 3/2 \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\}$$

$$\begin{pmatrix} -7/2 \\ 3/2 \\ 0 \end{pmatrix} + \underbrace{\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle}_{Sol(Ax=0)}$$

- $b \neq 1, b \neq 3$ il sistema non ha soluzioni

Es: Provare a rifarlo, scrivendo $a = s+1$ $b = t-1$ e discuterlo in s e t.

Esercizio Siano dati i sottospazi di K^5

$$U_1 = \left\langle \underbrace{\begin{pmatrix}1\\0\\1\\-1\\0\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}1\\2\\-4\\-1\\-2\end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix}0\\1\\2\\0\\1\end{pmatrix}}_{u_3} \right\rangle \qquad U_2 : \begin{cases} 2x_1 - x_2 - 2x_3 = 0 \\ x_5 = 0 \end{cases}$$

$$U_3 : \begin{cases} x_5 = 0 \\ x_3 = 0 \\ x_2 + 2x_4 = 0 \\ -x_2 + 3x_3 - 2x_4 + x_5 = 0 \end{cases}$$

Determinare dim., basi, eq. cartesiane di U_1, U_2, U_3, $U_1 \cap U_2$, $U_1 \cap U_2 \cap U_3$, $(U_1 + U_3) \cap U_2$, $U_1 + U_2$

$$U_1 = \left\langle \underbrace{\begin{pmatrix}1\\0\\1\\-1\\0\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}0\\1\\2\\0\\1\end{pmatrix}}_{u_3}, \cancel{\underbrace{\begin{pmatrix}0\\2\\-5\\0\\-2\end{pmatrix}}_{u_2-u_1}} \right\rangle \leftrightarrow \underbrace{\begin{pmatrix}0\\0\\-9\\0\\-4\end{pmatrix}}_{u_2-u_1-2u_3} \to$$ sono in scalette dunque lin. indip.

una base di U_1: $\left\{ \begin{pmatrix}1\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-9\\0\\-4\end{pmatrix} \right\}$ $\dim U_1 = 3$

Ma anche i 3 vettori u_1, u_2, u_3 formano base.

Eq. parametriche:

$$\begin{pmatrix}x_1\\x_2\\ \vdots \\x_5\end{pmatrix} = \alpha \begin{pmatrix}1\\0\\1\\-1\\0\end{pmatrix} + \beta \begin{pmatrix}0\\1\\2\\0\\1\end{pmatrix} + \gamma \begin{pmatrix}0\\0\\-9\\0\\-4\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\ -9\gamma+\alpha+2\beta \\ -\alpha \\ \beta - 4\gamma\end{pmatrix}$$

$$\begin{cases} x_1 = \alpha \\ x_2 = \beta \\ x_3 = -9\gamma + \alpha + 2\beta \\ x_4 = -\alpha \\ x_5 = \beta - 4\gamma \end{cases} \qquad \begin{cases} \alpha = x_1 \\ \beta = x_2 \\ \cancel{9}\gamma = \frac{x_1 + 2x_2 - x_3}{9} \\ x_4 = -x_1 \\ 9x_5 = 9x_2 - \cancel{9}\frac{4}{\cancel{9}}(x_1 + 2x_2 - x_3) \end{cases} \qquad \begin{cases} \cancel{x_1 = \alpha} \\ \cancel{x_2 = \beta} \\ x_1 + x_4 = 0 \\ \cancel{\gamma = \frac{x_1 + 2x_2 - x_3}{9}} \\ 4x_1 - x_2 - 4x_3 + 9x_5 = 0 \end{cases}$$

Verifica

$$\begin{cases} x_1 + x_4 = 0 \\ 4x_1 - x_2 - 4x_3 + 9x_5 = 0 \end{cases}$$ eq. cartes. di U_1

Studio U_2

$$\left(\begin{array}{ccccc|c} 2 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

x_1 x_5

rango 2 — x_1, x_5 indet. $x_2\ x_3\ x_4$ parametri

$\dim U_2 = 5 - 2 = 3$ eq. cartesiane di U_2 date nel testo,

$U_2 : Sol(Ax=0) \quad A = \begin{pmatrix} 2 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} a/2+b \\ a \\ b \\ c \\ 0 \end{pmatrix}, a,b,c \in \mathbb{R} \right\} = U_2 = \left\langle \begin{pmatrix} \cancel{1/2}\ 1 \\ \cancel{1}\ 2 \\ 0\ 0 \\ 0\ 0 \\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

verificare che siano soluzioni del sistema!

$\underbrace{\qquad\qquad\qquad}_{\text{mi aspetto formino base.}}$

$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \underset{2^\circ - 1^\circ}{\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ sono in scaletta dall'alto (sono in scaletta dal basso)

base di U_2

Studio U_3: $\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \end{array}\right) \underset{\sim}{\text{riordino}} \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$

$\underset{\substack{II - 3III \\ II_{new} - IV}}{\overset{II+I}{\sim}} \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & \cancel{3}\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{\text{riordino}}{\sim} \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

parametri

$rg\, A = 3 \Rightarrow$ il sistema ha una eq. superflua.

Eq. cartesiane di U_3: $\begin{cases} x_2 + 2x_4 = 0 \\ x_3 = 0 \\ x_5 = 0 \end{cases}$

$\Rightarrow \dim U_3 = 5 - 3 = 2 \quad \left\{ \begin{pmatrix} a \\ -2b \\ 0 \\ b \\ 0 \end{pmatrix}, a,b \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$U_1 \cap U_2 \sim \begin{cases} x_1 + x_4 = 0 \\ 4x_1 - x_2 - 4x_3 + 9x_5 = 0 \\ 2x_1 - x_2 - 2x_3 = 0 \\ x_5 = 0 \end{cases}$ c... verifica!

$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & -1 & -4 & 0 & 9 & 0 \\ 2 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{\text{Esercizio}}{\rightsquigarrow} \quad U_1 \cap U_2 = \left\langle \underset{w}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}} \right\rangle$

7 eq. :(

$U_1 \cap U_2 \cap U_3 =$ ragiono sulle dim. $\begin{cases} 1 \Leftrightarrow U_1 \cap U_2 \leq U_3 \Leftrightarrow w \in U_3 \\ 0 \end{cases}$

$\overset{\wedge|}{U_1 \cap U_2}$

? w è soluzione delle eq. di U_3? $\boxed{NO}$ dunque

- Studiare $(U_1 + U_3) \cap U_2$

$$U_1 + U_3 = \left\langle \begin{pmatrix}1\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\-5\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\9\\0\\4\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\0\\1\\0\end{pmatrix} \right\rangle$$

$$\left\langle \underset{v_1 - v_4}{\begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}}, \underset{v_2+v_5}{\begin{pmatrix}0\\0\\-5\\1\\-2\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}0\\0\\9\\0\\4\end{pmatrix}}, \underset{v_4}{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}}, \underset{v_5}{\begin{pmatrix}0\\-2\\0\\1\\0\end{pmatrix}} \right\rangle$$

$$w_1 = v_2 + v_5 + 5(v_1 - v_4) \qquad v_3 - 9(v_1 - v_4) = w_2$$

$$w_1 = \begin{pmatrix}0\\0\\0\\-4\\-2\end{pmatrix} \qquad w_2 = \begin{pmatrix}0\\0\\0\\9\\4\end{pmatrix}$$

$$\underset{v_4}{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}} \quad \underset{-v_5}{\begin{pmatrix}0\\2\\0\\-1\\0\end{pmatrix}} \quad \underset{v_1 - v_4}{\begin{pmatrix}0\\0\\1\\\cancel{-1}\\0\end{pmatrix}} \quad \underset{\frac{w_1}{-2}}{\begin{pmatrix}0\\0\\0\\2\\1\end{pmatrix}} \quad \underset{w_2 - \frac{9}{2}\left(\frac{w_1}{-2}\right)}{\begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\end{pmatrix}}$$

$\Rightarrow$ 5 l. ind. e generatori

$$\dim(U_1 + U_3) = 5 \implies U_1 + U_3 = K^5$$

$$(U_1 + U_3) \cap U_2 = U_2.$$

quale legame ?

Cosa succede se calcolo $U_1 \cap U_2 = ?$ $\quad U_3 \cap U_2 = ?$

- Osservo che le equazioni cambiano se cambio base in cui considero le coordinate (sopra ho sempre considerato le coord rispetto alla base canonica).

base di $\mathbb{R}^5$

$$\underbrace{\underset{}{v_1 =}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, v_2 = \begin{pmatrix}0\\-2\\0\\1\\0\end{pmatrix}}_{\text{base } U_3}, \underbrace{v_3 = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, v_4 = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}}_{\in U_2}, \; v_5 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \to \notin U_3 + U_2$$

eq. cartesiane di U_3 in questa base $\mathcal{V}$

$U_3 = \langle v_1, v_2 \rangle$

$$\alpha_v : K^5 \longrightarrow K^5 \quad (\text{inversa } v \longmapsto \text{le sue coordinate}$$

$$\begin{pmatrix}a_1\\\vdots\\a_5\end{pmatrix} \longmapsto \sum a_i v_i$$

$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$ sono le coord. di v_1, risp. di v_2, nella base v_i

$v_1 = 1v_1 + 0v_2 + 0v_3 + 0v_4 + 0v_5$

$v_2 = 0v_1 + 1v_2 + 0v_3 + 0v_4 + 0v_5$

A U_3 corrisponde il sottospazio U_3' generato dalle coord. di v_1, v_2

$<\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}>$. Le eq. di U_3 nella base $\mathcal{V}$ sono le eq. di U_3'

ossia $\begin{cases} x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{cases}$

Trovare le eq. di U_2 risp. alla nuova base