Theorem (Bolzano) $f : I \to \mathbb{R}$, I interval.

$x_1, x_2 \in I$ s. that $x_1 < x_2$

$f(x_1) < 0$ $\quad f(x_2) > 0$

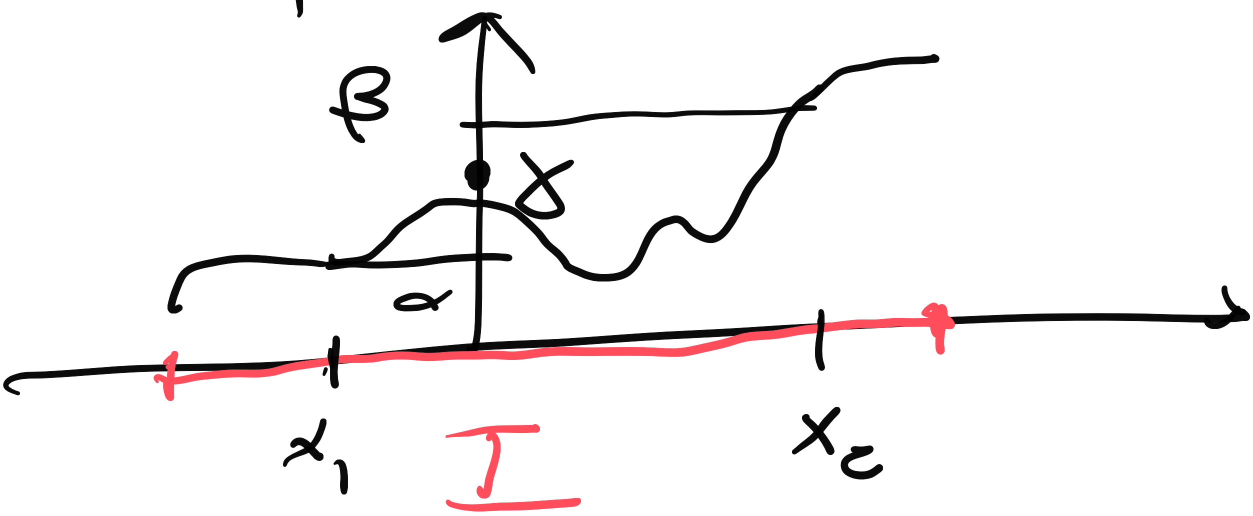
$$\Longrightarrow \exists\, \bar{x} \in \,]x_1, x_2[\, \text{ s.t. } f(\bar{x}) = 0$$

Corollary (Intermediate value theorem)

$f : I \to \mathbb{R}$ CONTINUOUS $\quad x_1, x_2 \in I$

$x_1 < x_2$ $\quad f(x_1) = \alpha \quad f(x_2) = \beta$

$\alpha < \beta$. If $\gamma \in \,]\alpha, \beta[$



Then $\exists\, \bar{x}$ s.t. $f(\bar{x}) = \gamma$

<u>Proof</u>: $g : I \longrightarrow \mathbb{R}$

$$g(x) := f(x) - \gamma$$

$$g(x_1) = f(x_1) - \gamma = \alpha - \gamma < 0$$

$$g(x_2) = f(x_2) - \gamma = \beta - \gamma > 0$$

by Bolzano th.

$\exists\, \bar{x}$ s.t. $g(\bar{x}) = 0$

$$0 = g(\bar{x}) = f(\bar{x}) - \gamma$$

$$\Longleftrightarrow \quad f(\bar{x}) = \gamma$$

q.e.d.

<u>Corollary</u>. $f : [a, b] \longrightarrow \mathbb{R}$ continuous. Then

$$\boxed{f([a, b]) = [m, M]}$$

where $m = \min f$, $M = \max f$

(existing because Weierstrass Th.)

Exercise: $f : [a, b] \to \mathbb{R}$

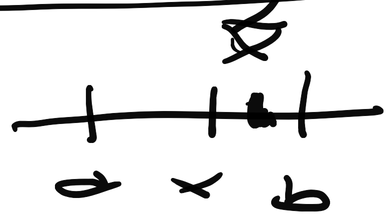
f injective and continuous, $f(a) < f(b)$.

Let $x \in]a, b[$. Then

$$f(a) < f(x) < f(b)$$

$f(x) \stackrel{?}{=} f(a)$, $f(x) \stackrel{?}{=} f(b)$ } no by injectivity

$$f(x) \stackrel{?}{<} f(a) < f(b)$$



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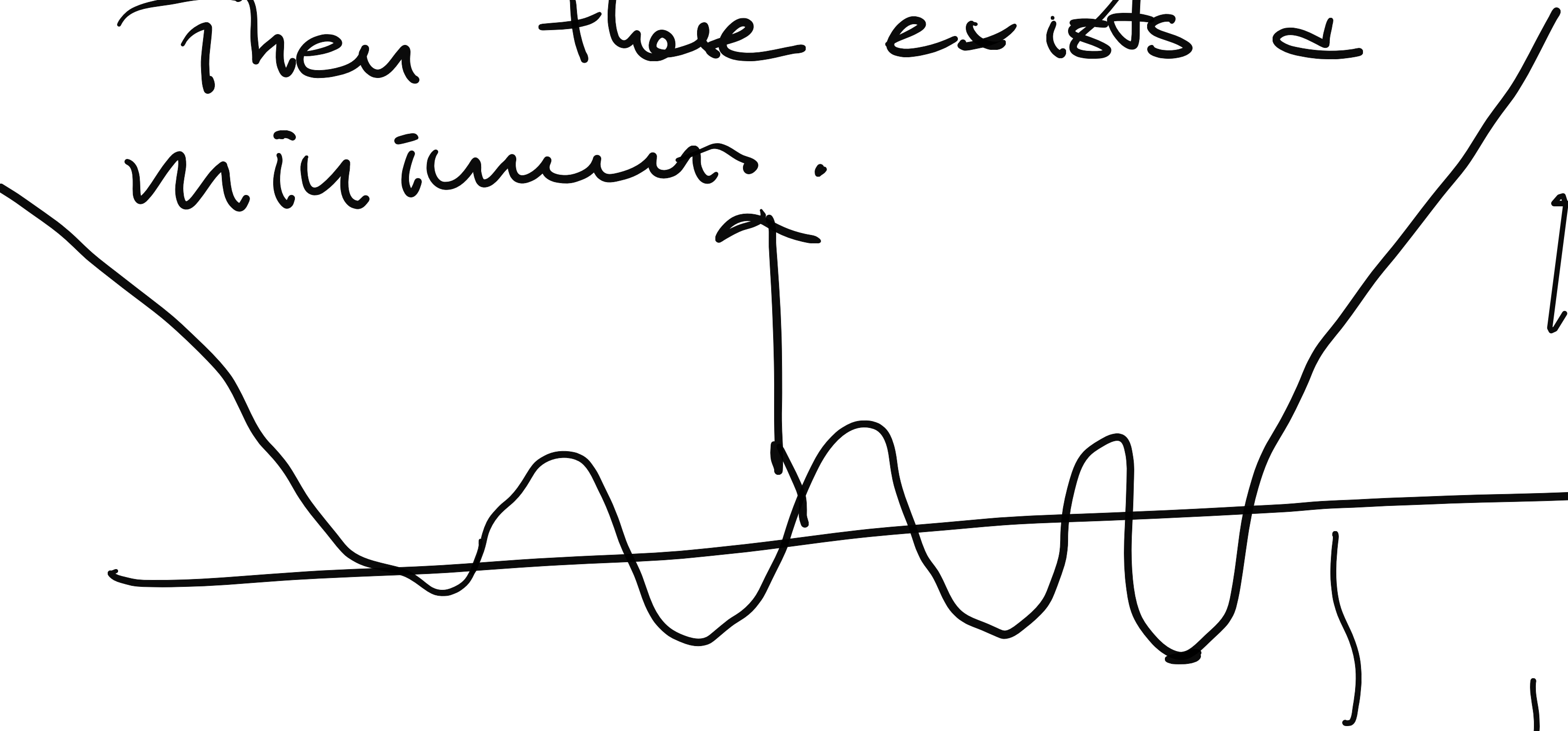
$\Rightarrow \exists \bar{x}$ s.t. $f(\bar{x}) = \gamma = f(a)$

Exercise $f : \mathbb{R} \to \mathbb{R}$ continuous.

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = +\infty$$

Then there exists a minimum.



By contradiction there is no minimum

For every $n \in \mathbb{N}$

$\exists\ x_n \in \mathbb{R}$ s.t. $f(x_n) \leq -n$

$c = \inf(f(\mathbb{R}))$

$c \in \mathbb{R}$ $\forall n \in \mathbb{N}$

$\exists y_n \in f(\mathbb{R})$ s.t.

$$c \le y_n < c + \frac{1}{n}$$
$$y_n = f(x_n)$$

if (x_n) is unbounded

there is a subsequence

$x_{n_k} \to +\infty$ (or $-\infty$)

$$\lim_{k \to +\infty} f(x_{n_k}) = +\infty$$
$$f(x_{n_k}) = y_{n_k} \to c \in \mathbb{R}$$

$\inf(f(\mathbb{R})) = c = -\infty$

$\forall n \; \exists \; x_n$ s.t.

(*) $f(x_n) < -n$

if (x_n) is unbounded

$\exists \; x_{n_k} \to +\infty$ (or $-\infty$)

$\lim_{k\to\infty} f(x_{n_k}) \to -\infty$

by hypothesis $\lim f(x_{n_k}) = +\infty$

Finish the exercise with showing that also (x_n) bounded is a contradiction (Weierstrass Th)

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Exercise: $f, g : \mathbb{R} \to \mathbb{R}$

1) Find f and g s.t.
- f is continuous
- g is not continuous

but $g \circ f$ is continuous.

2) Find f and g both not continuous, but $g \circ f$ is continuous.

Exercise: $f(x) = 3x^3 - 8x^2 + x + 3$

Show that there are 3 distinct solutions of $f(x) = 0$
$x_1, x_2, x_3 \in \mathbb{R}$, with
$x_1 \in]-\infty, 0[$, $x_2 \in]0, 1[$,
$x_3 \in]1, +\infty[$.

$f(0) = 3$

$f(-1) = -3 - 8 - 1 + 3 = -9$

$f(0) > 0 \quad f(-1) < 0$

$\Longrightarrow x_1 \in]-1, 0[$

Bolzano th

$f(x_1) = 0$

$f(1) = 3 - 8 + 1 + 3 =$

$= -1.$

$\Longrightarrow \exists \; x_2 \in]0, 1[$

Bolz th.

s.t. $f(x_2) = 0$

$f(2) = 24 - 32 + 2 + 3 = 0$

$= -8 + 2 + 3 = -3 < 0$

$f(3) = 243 - 72 + 3 + 3 > 0$

$$\Longrightarrow \quad x_3 \in \,]1, 3[$$

q.e.d

Exercise

$$p(x) := a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0$$

$a_n \neq 0$, a polynomial of degree n, n odd.

Show that the equation $p(x) = 0$ has (at least) one solution.

Suppose $a_n > 0$

$$\lim_{x \to +\infty} a_n x^n + a_{n-1} x^{n-1} \dots + a_0 =$$

$$= x^n \left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right)$$

$x^n \to +\infty$, and the bracket $\to a_n$

$= +\infty$

If $a_n < 0$

$$\lim_{x \to +\infty} p(x) = -\infty$$

If $a_n > 0$ $\quad \lim_{x \to -\infty} p(x) = -\infty$

$a_n < 0$ $\quad \lim_{x \to -\infty} p(x) = +\infty$

If $a_n > 0$

Since $p(x) \xrightarrow[x \to +\infty]{} +\infty$

(*) There is M s.t. $\forall x > M$

$p(x) > 1$

Since $p(x) \xrightarrow[x \to -\infty]{} -\infty$

There is K s.t.

(**) $\forall x < K$ $p(x) < -1$

Choos x_1 according to (*) and x_2 according (**)

$P(x_1) > 1 > 0$

$P(x_2) < -1 < 0$

Bolza $\Longrightarrow$ $\exists$ $\bar{x}$ in the interval of extremes x_1, x_2 such that

$P(\bar{x}) = 0$

Another method:

I know I can factorize

$p(x) = p_1(x) \cdot p_2(x) \cdot \ldots \cdot p_k(x)$

all p_i being of degree 1 or 2

Since p has odd degree there must be at least one p_i having degree 1

$p_i = (ax + b)$

$p(x) = (ax + b)(q(x))$

if $\tilde{x} = -\frac{b}{a}$

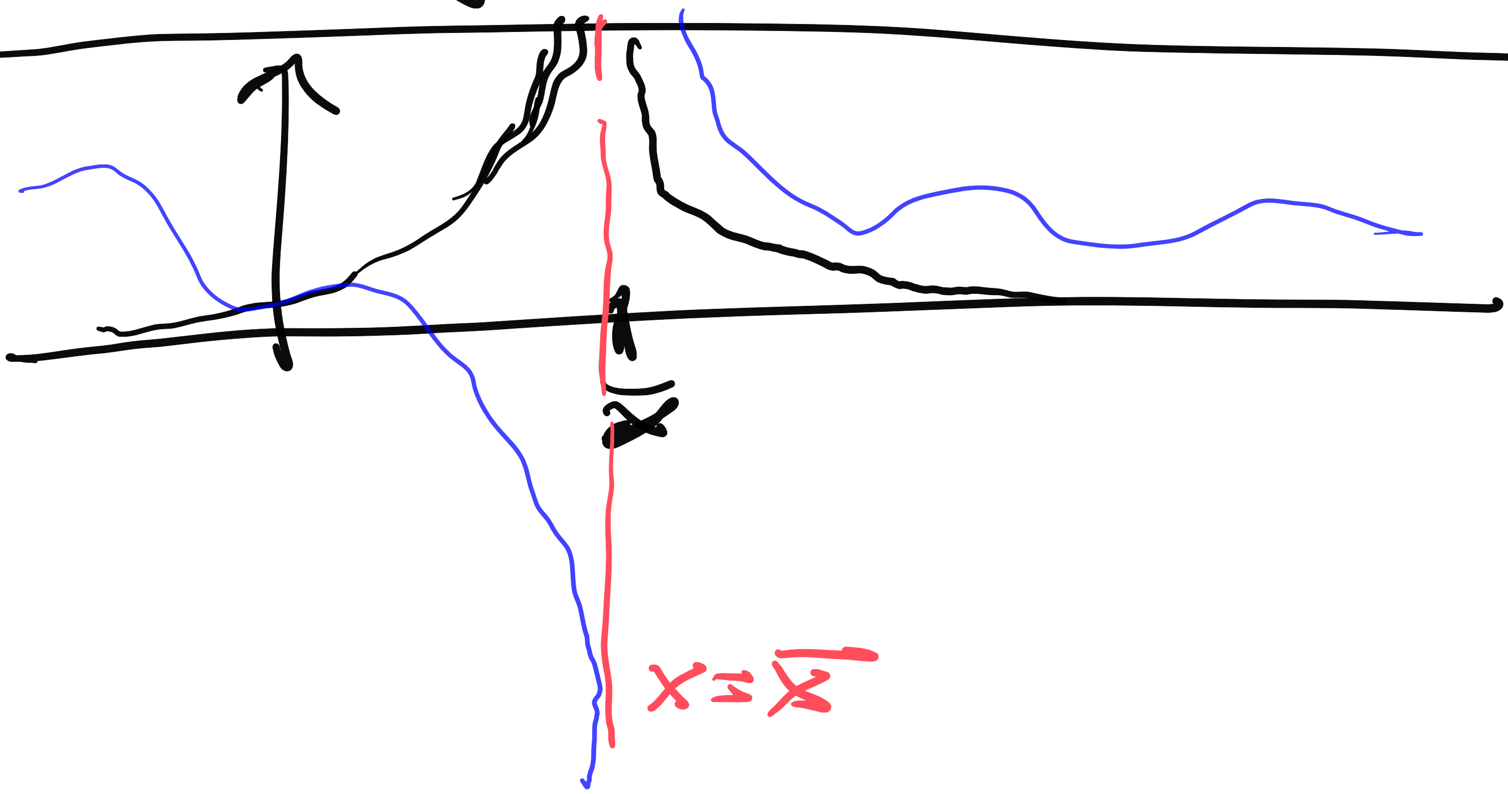
$p(\tilde{x}) = 0 \cdot q(x) = 0$

$f : D \longrightarrow \mathbb{R}$

Simplest functions are the linear ones, that

$$r(x) = mx + q,$$

whose graphs are lines



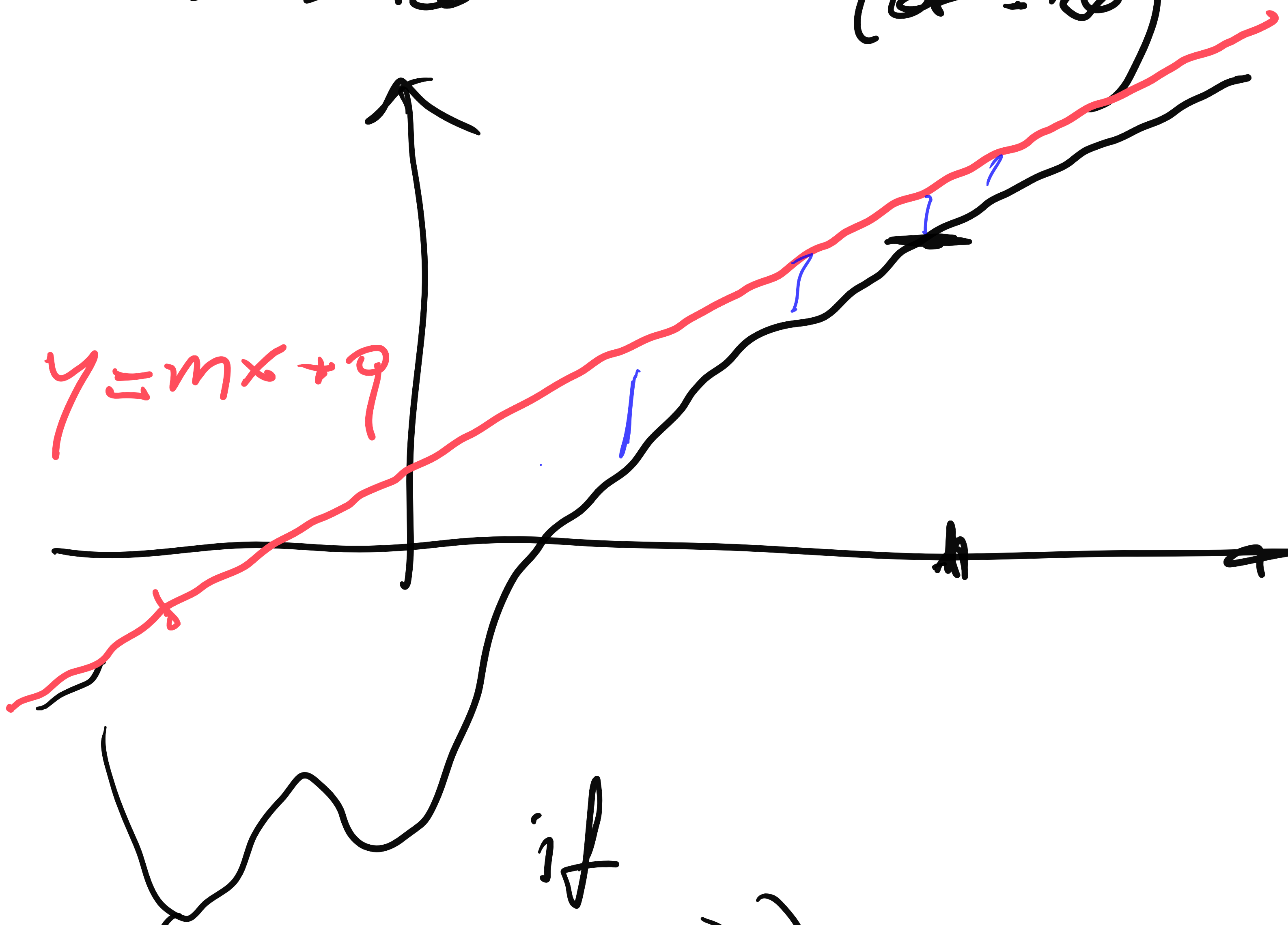
The line $x = \bar{x}$ is a vertical asymptote of f if $\lim_{x \to \bar{x}} f(x) = \pm\infty$

or $\lim_{x \to \bar{x}^{\pm}} f(x) = \pm\infty$

If $\lim_{x \to +\infty} f(x) = \underline{l} \in \mathbb{R}$

we say that $f(x)$ has the horizontal asymptote for $x \to +\infty$ $y = l$

$$\lim_{x \to +\infty} f(x) = +\infty$$
$(\text{or } -\infty)$



if

$$\lim_{x \to +\infty} \left(f(x) - (mx+q)\right) = 0 \quad (*)$$

$y = mx + q$ is called asymptote of f for $x \to +\infty$

Determine m and q

By (*)

$$\lim_{x \to +\infty} \frac{f(x) - mx - q}{x} = 0$$

$$\Downarrow$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} - m = 0$$

$$\Rightarrow \quad m = \lim \frac{f(x)}{x}$$

From (*)

$$q = \lim_{x \to +\infty} f(x) - mx$$

$$f(x) = x + x^2$$

$$\lim \frac{x + \cancel{x^2} - \cancel{x^2}}{\sqrt{x+x^2} + x} = \frac{1}{2}$$

$y = x + \frac{1}{2}$ is the asymtote

$f(x) = x + \log x$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \frac{x}{x} + \frac{\log x}{x} \to 1$$

$m = 1$

$$q = \lim_{x \to +\infty} f(x) - x = x + \log x - x$$

$= \lim \log x = +\infty$