1) QUESTIONS ABOUT PREVIOUS EXERCISES
2) LIMITS
(3) REVIEW EXERCISES / CONTINUITY)

2) $\lim_{x\to+\infty} \dfrac{3x^2-(e^{x^2}-1)\sin^2 x}{x^3+e^{\alpha x}}$ $\alpha\in\mathbb{R}$

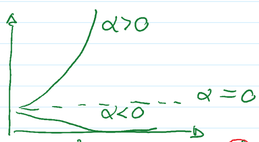
N: $\lim_{x\to+\infty} 3x^2-(e^{x^2}-1)\sin^2 x \approx \lim_{x\to+\infty} 3x^2 - e^{x^2}\sin^2 x$

$\lim_{x\to+\infty} \dfrac{3x^2}{e^{x^2}\sin^2 x} = \lim_{x\to+\infty} 3x^2 e^{-x^2}\dfrac{1}{\sin^2 x} = \left[\dfrac{0}{0}\right]$ $\sin^2 x \to [0,1]$

$3x^2 = o\left(e^{x^2}\sin^2 x\right)$

$\approx \lim_{x\to+\infty} -e^{x^2}\sin^2 x$ does not exist because $\lim_{x\to+\infty}\sin^2 x$ does not exist

$\frac{N}{D}$: $\lim_{x\to\infty} \dfrac{-e^{x^2}\sin^2 x}{x^3+e^{\alpha x}}$



$\alpha=0$ $\lim_{x\to\infty} -\dfrac{e^{x^2}}{x^3+1}\sin^2 x \approx \lim_{x\to\infty} -\dfrac{e^{x^2}}{x^3}\sin^2 x$ $\nexists$

$\alpha<0$ $\lim_{x\to\infty} -\dfrac{e^{x^2}}{x^3+o(x^3)}\sin^2 x$ $\nexists$

$\alpha>0$ $\lim_{x\to+\infty} -\dfrac{e^{x^2}}{x^2+e^{\alpha x}}\sin^2 x = \lim_{x\to+\infty} -e^{x^2-\alpha x}\sin^2 x \approx \lim_{x\to\infty} -e^{x^2}\sin^2 x$ $\nexists$

$\to x^2 = o(e^x)$ $\alpha x = o(x^2)$

$\nexists \ \forall\alpha\in\mathbb{R}$

5) $\lim_{x\to+\infty} \dfrac{x^3-(3+\cos x)^x}{3^{-\alpha x}+2x^3}$

N: $x^3-(3+\cos x)^x = f(x)$ $\quad \cos x \to [-1,1]$ $\quad x^3-2^x \le f(x) \le x^3-4^x$ $\quad x^3 = o(a^x)$ $a>1$

$\lim -\infty$ $\quad \lim -\infty$

$\lim_{x\to+\infty} x^3-(3+\cos x)^x = -\infty$

$\frac{N}{D}$: $\lim_{x\to+\infty} \dfrac{-(3+\cos x)^x}{2x^3+3^{-\alpha x}}$

$\alpha=0 \vee \alpha>0$ $\quad 3^{-\alpha x} = o(x^3) \rightsquigarrow \lim_{x\to+\infty} -\dfrac{(3+\cos x)^x}{2x^3} = -\infty$

$\alpha\geq 0$ $\boxed{-\infty}$

$\alpha<0$ $\quad x^3 = o(3^{-\alpha x}) \rightsquigarrow \lim_{x\to+\infty} -\dfrac{(3+\cos x)^x}{3^{-\alpha x}} = \lim_{x\to+\infty} -\left(\dfrac{3+\cos x}{3^{-\alpha}}\right)^x$

$f(x) = \dfrac{3+\cos x}{3^{-\alpha}} \geq 1+\varepsilon$, $\varepsilon>0, \forall x$ $\quad \lim_{x\to+\infty} f(x) \geq \lim_{x\to+\infty} (1+\varepsilon)^x = +\infty$

$f(x) = \dfrac{3+\cos x}{3^{-\alpha}} \leq 1-\varepsilon$, $\varepsilon>0, \forall x$ $\quad \lim_{x\to+\infty} (1-\varepsilon)^x = 0 \geq \lim_{x\to+\infty} f(x)$

$f(x) = \frac{3+\cos x}{3^{-\alpha}} \leq 1-\varepsilon$, $\varepsilon > 0, \forall x$ $\lim_{x\to+\infty} (1-\varepsilon)^x = 0 \geq \lim_{x\to+\infty} f(x)$

$3+\cos x \geq (1+\varepsilon)3^{-\alpha} > \underbrace{3^{-\alpha}}_{<2}$ $\quad 3^{1+\alpha} + \frac{\cos x}{3^{-\alpha}} > 1$

$[2\ 4]$

$\Rightarrow 3^{-\alpha} < 2$ $\quad \alpha > -\log_3 2 \quad -\infty$

because we found another function that constrains the one we are studying to these limits

$\underset{[2\ 4]}{3+\cos x} \leq (1-\varepsilon) 3^{-\alpha} < \underbrace{3^{-\alpha}}_{>4} \Longrightarrow 3^{-\alpha} > 4$ $\quad \alpha < -\log_3 4 \quad 0$

$-\log_3 4 \leq \alpha \leq -\log_3 2$ $\quad \nexists$

we can find 2 subsequences of values for x that leads to two different limits [e.g. when $\cos x_{n_1} = 1$ and $\cos x_{n_2} = -1$ $\lim f(x_{n1}) \neq \lim f(x_{n2})$]

$\to$ the limit of the sequence $\lim_{n\to+\infty} f(x_n) \nexists$ thus also the limit of our function $\nexists$

1) $\lim_{x\to+\infty} \frac{e^{-\frac{1}{x}}(2x^2 - x^{1-\alpha}) + 3x^{\alpha}\cos x}{x^2}$

N: $\underline{2x^2 e^{-\frac{1}{x}}} - x^{1-\alpha} e^{-\frac{1}{x}} + 3x^{\alpha}\cos x$

$\lim_{x\to+\infty} e^{-\frac{1}{x}}\left(2 - x^{-1-\alpha}\right) + 3x^{\alpha-2}\cos x$

$\alpha > 2$ $\quad x^{\alpha-2} \to +\infty$ $\quad \lim_{x\to+\infty} 3x^{\alpha-2}\cos x \ \nexists$

$x^{-1-\alpha} \to 0$, $e^{-\frac{1}{x}} \to 1$ $\quad \lim_{x\to+\infty} e^{-\frac{1}{x}}(2 - x^{-1-\alpha}) = 2$

$x^{-1-\alpha} = o(x^{-3})$

$\alpha \geq 2 \ \nexists$

$\alpha = 2$ $\quad \lim_{x\to+\infty} 3\cos x \ \nexists$

$\alpha < 2$ $\quad x^{\alpha-2} \to 0$ $\quad \lim_{x\to+\infty} 3x^{\alpha-2}\cos x = 0$

$-1 < \alpha < 2$ $\quad x^{-1-\alpha} \to 0$ $\quad \lim_{x\to\infty} e^{-\frac{1}{x}} \cdot 2 = 2$

$-1-\alpha < 0 \quad \alpha > -1$

$-1 < \alpha < 2 \ \boxed{2}$

$\alpha = -1$ $\quad x^{-1-\alpha} = 1$ $\quad \lim_{x\to\infty} e^{-\frac{1}{x}}(2-1) = 1$

$\alpha = -1 \ \boxed{1}$

$\alpha < -1$ $\quad x^{-1-\alpha} \to +\infty$ $\quad \lim_{x\to\infty} -x^{-1-\alpha} = -\infty$

$\alpha < -1 \ \boxed{-\infty}$

11) $\lim_{x\to 0^+} \frac{3\operatorname{arctg} x - 3\alpha^2 x - \alpha x^3}{3\sin^3 x - x^4}$

D: $3\sin^3 x - x^4 = 3x^3 + o(x^3) - x^4 = 3x^3 + o(x^3)$

N: $3\left(x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)\right) - 3\alpha^2 x - \alpha x^3$

$3(1-\alpha^2)x - (\alpha+1)x^3 + \frac{3}{5}x^5 + o(x^5)$

$\frac{N}{D}$: $\lim_{x\to 0^+} \frac{3(1-\alpha^2)x - (\alpha+1)x^3 + \frac{3}{5}x^5 + o(x^5)}{3x^3 + o(x^3)}$

$\alpha < -1 \vee \alpha > 1$ $\quad (1-\alpha^2) < 0$ $\quad \lim_{x\to 0^+} \frac{\cancel{3}(1-\alpha^2)x + o(x)}{\cancel{3}x^3 + o(x^3)}$

$\lim_{x\to 0^+} (1-\alpha^2)x^{-2} = \boxed{-\infty}$

$-1 < \alpha < 1$ $\quad (1-\alpha^2) > 0$ $\quad \lim_{x\to 0^+} (1-\alpha^2)x^{-2} = \boxed{+\infty}$

$\alpha = 1$ $\quad \lim \ \frac{-2x^3 + o(x^3)}{\ldots} = \boxed{-\frac{2}{3}}$

$\boxed{-1<\alpha<1}$ $(1-\alpha^2)>0$ $\lim_{x\to 0^+} (1-\alpha^2)x^{-2} = \boxed{+\infty}$

$\alpha=1$ $\lim_{x\to 0^+} \frac{-2x^3+o(x^3)}{3x^3+o(x^3)} = \boxed{-\frac{2}{3}}$

$\alpha=-1$ $\lim_{x\to 0^+} \frac{3/5\,x^5+o(x^5)}{3x^3+o(x^3)} = \boxed{0}$

7) $\lim_{x\to 0} \frac{2^{x^2-x}-2^x+2\log(2)x+x^3\sin(\frac{1}{x})}{\sin(\alpha x^2)+(\cos x-1)^2+e^{-3/x^2}}$

N: $e^{(x^2-x)\log 2} - e^{x\log 2} + 2\log(2)x + \underbrace{x^3\sin\left(\frac{1}{x}\right)}_{o(x^2)}$

$\cancel{1}+(x^2-x)\log 2+\frac{(x^2-x)^2\log^2 2}{2}+o(x^2)\cancel{-1}-x\log 2-\frac{x^2\log^2 2}{2}+o(x^2)$ $+2\log 2\,x$

$-x\cancel{\log 2}+x^2\log 2+\cancel{\frac{x^2\log^2 2}{2}}+o(x^2)\cancel{-x\log 2}-\cancel{\frac{x^2\log^2 2}{2}}+o(x^2)+\cancel{2\log 2\,x}$

$x^2\log 2+o(x^2)$

D: $\sin(\alpha x^2)+(\cos x-1)^2+e^{-3/x^2}$

$\alpha x^2-\frac{(\alpha x^2)^3}{6}+o(\alpha x^6)+\left(\cancel{1}-\frac{x^2}{2}\cancel{-1}+o(x^2)\right)^2+e^{-3/x^2}$ $\to o(x^4)$

$\alpha x^2-\frac{\alpha^3}{6}x^6+o(\alpha x^6)+\frac{x^4}{4}+o(x^4)$

$\frac{N}{D}$: $\lim_{x\to 0} \frac{x^2\log 2+o(x^2)}{\alpha x^2-\frac{\alpha^3}{6}x^6+o(\alpha x^6)+\frac{x^4}{4}+o(x^4)}$

$\alpha\neq 0$ $\lim_{x\to 0}\frac{x^2\log 2+o(x^2)}{\alpha x^2+o(x^2)} = \boxed{\frac{\log 2}{\alpha}}$

$\alpha=0$ $\lim_{x\to 0}\frac{x^2\log 2+o(x^2)}{\frac{x^4}{4}+o(x^4)} = \lim_{x\to 0}\frac{\log 2}{4}\frac{1}{x^2} = \boxed{+\infty}$

HINT TO SOLVE A PREVIOUS EXERCISE

3) $\lim_{x\to a^+}(e^x-e^a)^{\frac{1}{\log(\sin(x-a))}}$ we would like to have $\lim_{y\to 0^+}$ with $y=x-a$

$$(e^x-e^a)^{\frac{1}{\log(\sin(x-a))}} = e^{\frac{\log(e^x-e^a)}{\log(\sin(x-a))}} = e^{\frac{\log[(e^{x-a}-1)e^a]}{\log(\sin(x-a))}} = e^{a\frac{\log(e^{x-a}-1)}{\log(\sin(x-a))}}$$

PARAMETRIC LIMITS ($\alpha\in\mathbb{R}$)

1) $\lim_{x\to+\infty}\frac{e^{-\frac{1}{x}}(2x^2-x^{1-\alpha})+3x^\alpha\cos x}{x^2}$

2) $\lim_{x\to+\infty}\frac{3x^2-(e^{x^2}-1)\sin^2 x}{x^3+e^{\alpha x}}$

3) $\lim_{x\to+\infty}\frac{\cos x+x(\mathrm{sgn}(\sin x))}{x^\alpha}$

4) $\lim_{x\to+\infty}\frac{(x+\cos x)^\alpha}{\sin(1/x)}$

5) $\lim_{x\to+\infty}\frac{x^3-(3+\cos x)^x}{3^{-\alpha x}+2x^3}$

6) $\lim_{x\to+\infty}\left(x^{\frac{x+1}{x}}-x-\alpha\log x\right)$

7) $\displaystyle \lim_{x \to 0} \frac{2^{x^2-x} - 2^x + 2\ln(2)x + x^3 \sin\left(\frac{1}{x}\right)}{\sin(\alpha x^2) + (\cos x - 1)^2 + e^{-3/x^2}}$

8) $\displaystyle \lim_{x \to +\infty} \frac{\left(\frac{1}{x}\right)^{\frac{1}{x}} - 2e^{\frac{1}{x}} + \cos\left(\frac{1}{x}\right) + \frac{\alpha}{x} \log\left(\frac{1}{x}\right)}{\left(\sqrt{1+\sinh\left(\frac{1}{x}\right)} - \sqrt{1+\sin\left(\frac{1}{x}\right)}\right)^{1/3}}$

9) $\displaystyle \lim_{x \to 0} \frac{\sqrt{3}\,(e^x)^2 - 2\sqrt{3}x + 2x^2 - \sqrt{3}}{3 + \alpha x^2 - 3e^{x^2} + x^4 \cos\left(\frac{6}{x}\right)}$

10) $\displaystyle \lim_{x \to +\infty} \frac{x^\alpha - \alpha^x + \sin(x^2)}{6e^x + \sqrt{x^5 + 2x + 1}}$

11) $\displaystyle \lim_{x \to 0^+} \frac{3\,\mathrm{arctg}\, x - 3\alpha^2 x - \alpha x^3}{3\sin^3 x - x^4}$

12) $\displaystyle \lim_{x \to 0^+} \frac{3^{x+x^2} - e^x + (1-\log 3)\sin x + (1+\log 3)\, 3^{-\frac{1}{x}}}{(\alpha-2)x^2 + x^3 \log x}$

13) $\displaystyle \lim \frac{\frac{\alpha}{n^{3/2}} - 6\left(\frac{\alpha^2}{\sqrt{n}} - \sin\left(\frac{1}{\sqrt{n}}\right)\right) - \frac{2}{n^2}}{\left(1 + \frac{1}{n} - \frac{2}{n^3}\right)^{\sqrt{n}} - 1}$

14) $\displaystyle \lim_{n \to +\infty} \frac{1 + \alpha n^3 - n \sin n + n^4 \sin \frac{1}{n}}{\alpha n^3 + \log^4 n + \sqrt{n^3 + 1}}$

15) $\displaystyle \lim_{n \to +\infty} \frac{4^n - \alpha^n}{n^4 2^n + 4^n}$

16) $\displaystyle \lim_{n \to +\infty} \frac{1 - n^2 \log\left(1 + \frac{1}{n^2}\right) + \frac{\sin n}{n^3}}{n^\alpha\, \mathrm{arctg}\, \frac{1}{n}}$