Continuity

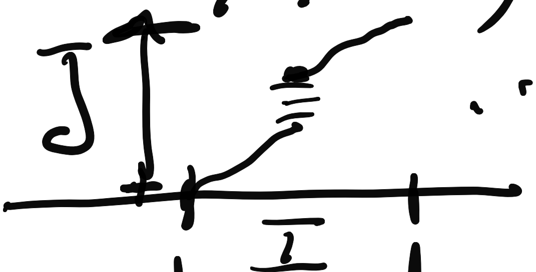
$f : D \to \mathbb{R}$, $\xi \in Acc(D) \cap D$

f continuos at ξ if $\lim_{x \to \xi} f(x) = f(\xi)$

- o.k. operations $f+g$, $f \cdot g$, $\frac{f}{g}$
- o.k composition
- $f : I \to J$, I, J intervals

f is monotonic and surjective, then it continuos on I ($\equiv$ at every $\xi \in I$)

surjectivity



monotonic but not surjective

not continuos

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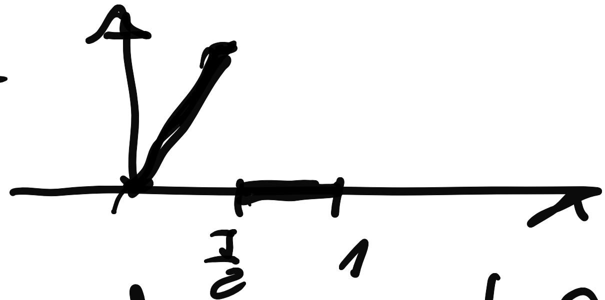
surjective but not monotonic

$f : [0, 1] \to [0, 1]$

$$f(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 0 & x \in]\frac{1}{2}, 0] \end{cases}$$



not continuous at $\xi = \frac{1}{2}$

$\sin : \mathbb{R} \longrightarrow [-1, 1]$

<u>Corollary:</u> $f : I \longrightarrow J$

f strictly monotonic

f surjective.

f is continuous

f^{-1} is continuous.

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x^{α} are continuous on
$\alpha \in \mathbb{R}$ $[0, +\infty[$

a^x are continuous

$\log_a$ " "

$\arcsin\ [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arccos$

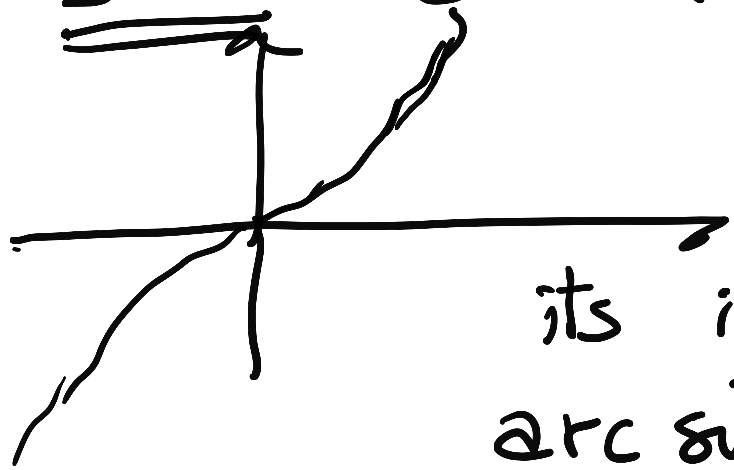
arctg are continuous

$$\sinh = \frac{e^x - e^{-x}}{2}$$

$$e^{-x} = g(h(x)) \quad \text{with } h(x) = -x, \; g(y) = e^y$$

$$\cosh = \frac{e^x + e^{-x}}{2}$$

sinh is increasing on $\mathbb{R}$



its inverse

arc sinh

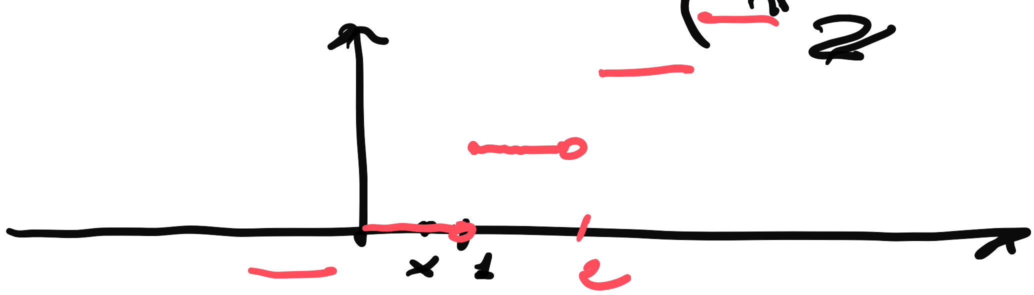
$$\text{arcsinh} = \sinh^{-1}$$

Exercise: find an expression of the inverse in terms of log.

$$y = \sinh(x)$$

$$x = \dots\dots$$

$$x \longmapsto [x] = \max\{n \in \mathbb{Z} : n \leq x\}$$



Not continuous on the points $x \in \mathbb{Z}$

Teorem (Bolzano) "Zero theorem"

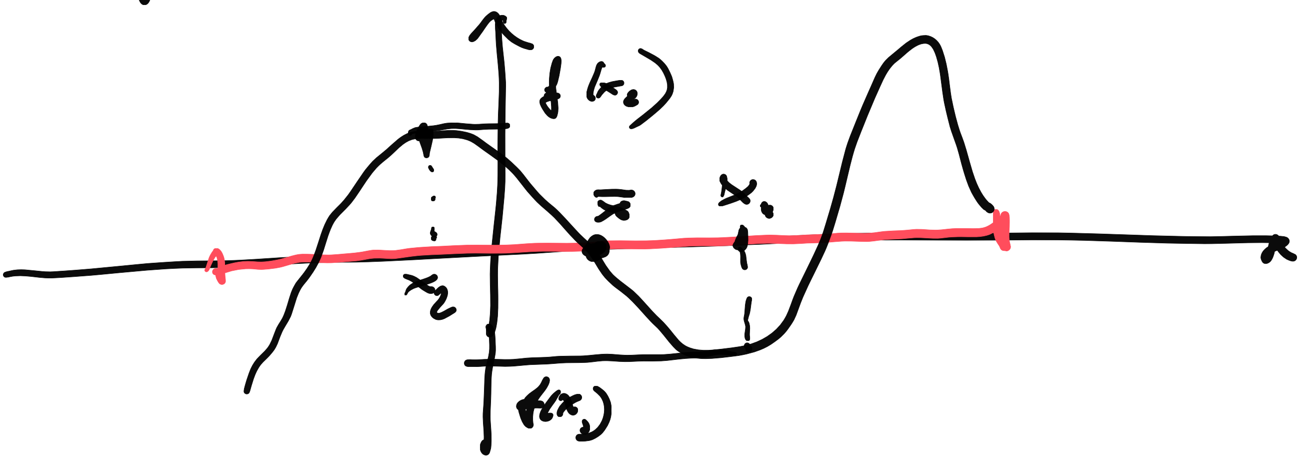
$f: I \longrightarrow \mathbb{R}$ $\quad I$ interval

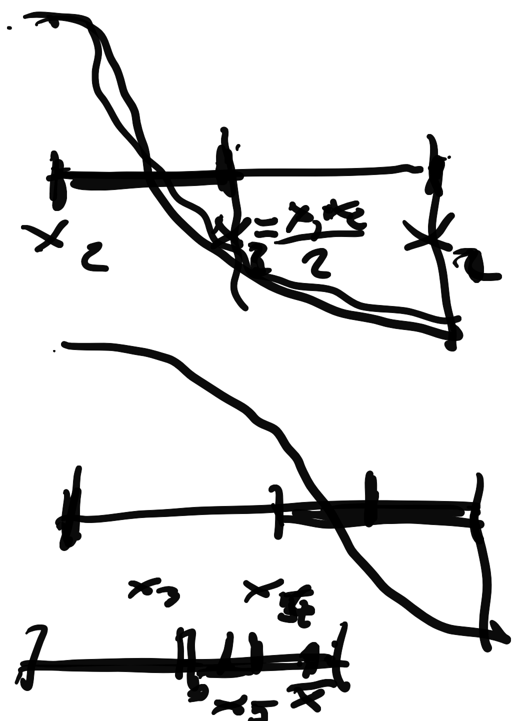
continuous

$x_1, x_2 \in I \quad f(x_1) < 0, \quad f(x_2) > 0$

$\exists \quad \bar{x} \in \,]x_1, x_2[\,$ s.t.

$f(\bar{x}) = 0$





Bolzano-Weierstrass Theorem: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence, let (x_n) be bounded i.e., that is $\{x_n, n\in\mathbb{N}\}$ is bounded.

Then there exists a subsequence $k\mapsto x_{n_k}$ which converges to some $\bar{x}\in\mathbb{R}$

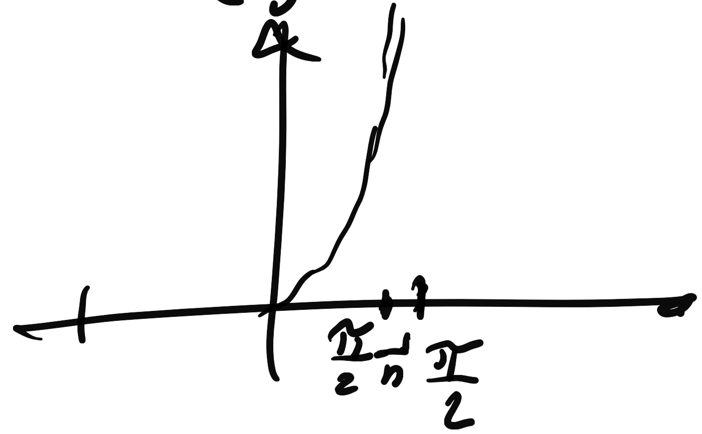
(Boundedness is important, but not necessary

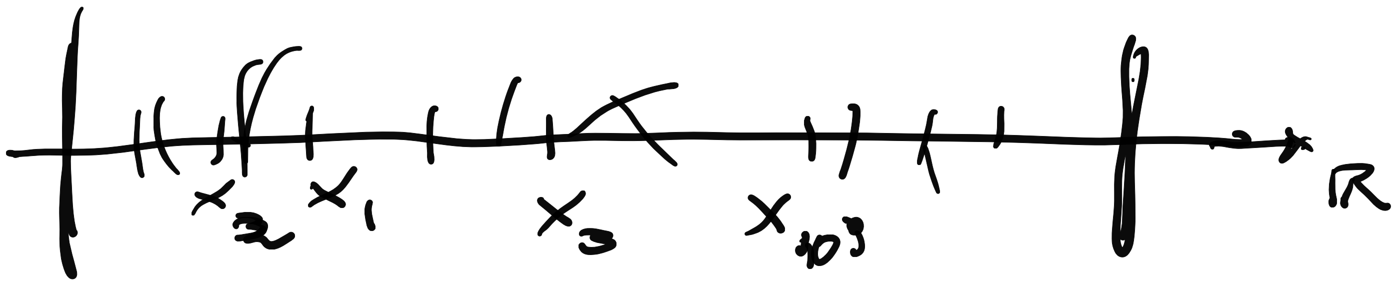
IMPORTANT: Every (x_n) $x_n\to+\infty$ has not convergent subsequence

NOT NECESSARY: $x_n=\begin{cases} n & n \text{ is odd} \\ 0 & n \text{ even} \end{cases}$ is bounded

$x_n=\sin n$ is bounded, $x_n=\frac{1}{n}$ is bounded

$x_n=(-1)^n$ is bounded, $x_n=\mathrm{tg}\left(\frac{\pi}{2}-\frac{1}{n}\right)$ not bounded





$x_n = (-1)^n$

minimum (singular) minima (plural) minimums

maximum (singular) maxima (plural) maximums.

<u>Def</u> $f : D \longrightarrow \mathbb{R}$

we say that $\bar{x} \in D$ is maximum (minimum) point $\underbrace{\text{if } f(\bar{x}) \geq f(x) \ \forall x \in D}$ $(\leq)$

$\Updownarrow$

$f(\bar{x}) = \max f(D)$

$(f(\bar{x}) = \min f(D))$

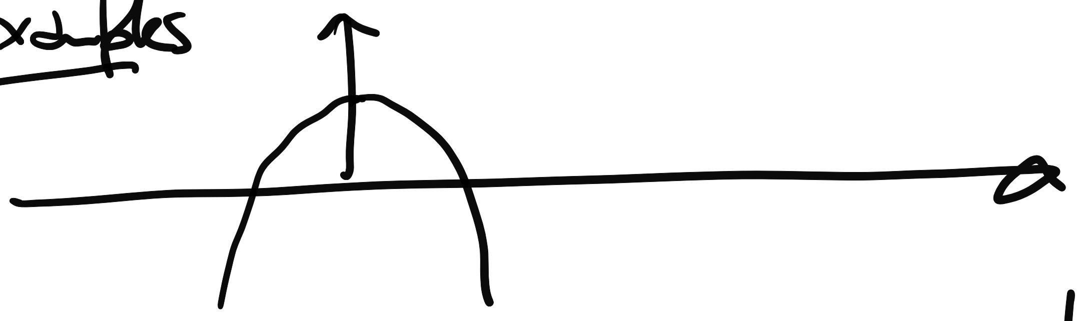
($f(\bar{x})$ is "the maximum" (minimum))

maximum $\equiv$ absolute maximum $\equiv$

$=$ global maximum

<u>Examples</u>

$f(x) = 1 - x^2$

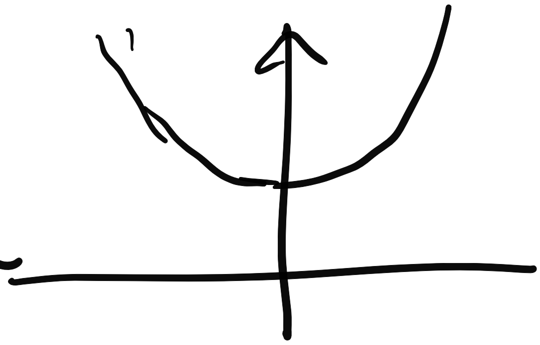
1) 

$\bar{x} = 0$ is a maximum point.

and $f(0) = 1$ is the maximum

there are <u>not minimum point</u>

2) $f(x) = 1 + x^2$



0 is a minimum point.

3) $f: \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$

$f(x) = tg\, x$

no maximum

no minimum

Theorem Weistrass: $f:[a,b] \to \mathbb{R}$ is continuous then

there exists (at least) a maximum point x_M

and " " (at least) a minimum point x_m

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1) importance of boundedness of the interval. $f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$

it has not a maximum.

2) importance of closedness

sin : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$

no maximum

3) importance of continuity

$f : [-1, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x} & x \in [-1, 1] \setminus \{0\} \\ 0 & x = 0 \end{cases}$$

continuous at all points $\xi \in [-1, 1] \setminus \{0\}$

not continuous in $\xi = 0$

has neither minimum nor maximum.

Proof of Weirstrass Theorem

I) $f([a,b]) = \{f(x),\ x \in [a,b]\}$ is bounded

"

upper bounded

&

lower bounded

Let us prove it is upper bounded.

By contradiction, suppose that $f([a,b])$ is not upper bounded

$\nexists\ n \in \mathbb{N}$ s.t. $f(x) \leq n \quad \forall x \in [a,b]$

$\Updownarrow$

$\forall n \in \mathbb{N} \quad \exists\ x_n \in [a,b]$ s.t. $\underline{f(x_n) > n}$

$(x_n)_{n\in\mathbb{N}}$ is a sequence $\underline{\subseteq [a,b]}$.

In particular, $(x_n)_{n\in\mathbb{N}}$ is bounded

$\underset{\text{subsequence}}{\exists}\ x_{n_k} \longrightarrow \bar{x} \in [a,b]$

Consider $\left(f(x_{n_k})\right)_{k\in\mathbb{N}}$

$\underbrace{f(x_{n_k})}$ $> n_k \Rightarrow \underline{f(x_{n_k}) \longrightarrow +\infty}$

and

$\underline{\text{by continuity}}$ $\underline{f(x_{n_k})} \longrightarrow f(\bar{x}) \in \mathbb{R}$

$\Longrightarrow \mathbb{R} \ni f(\bar{x}) = +\infty$

CONTRADICTION!

II) $f([a,b]) \subset \mathbb{R}$ is upper bounded

$\Longrightarrow f([a,b])$ has a supremum $S \in \mathbb{R}$

For every n, $S - \frac{1}{n} < S$

$\Longrightarrow y_n \in f([a,b])$ s.t. $\boxed{S - \frac{1}{n} < y_n \leq S}$

$y_n = f(x_n)$ for some $x_n \in [a,b]$

$(x_n) \subseteq [a,b]$, in particular (x_n) is bounded

by B-W Theorem, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such

$$x_{n_k} \longrightarrow \hat{x} \in [a,b].$$

$$f(x_{n_k}) \longrightarrow f(\hat{x})$$ by continuity

$$S - \frac{1}{n_k} \leq f(x_{n_k}) \leq S$$ by 2 policemen

$$\underbrace{S - \frac{1}{n_k}}_{\to S} \qquad f(x_{n_k}) \longrightarrow S$$

$\Longrightarrow S = f(\hat{x}) \in f([a,b])$

$\Longrightarrow S$ is a maximum, so $\hat{x}$ is a maximum point.