Lesson 19 - 10/11/2022

Ex 1 Let $f \in C^{\infty}(\mathbb{R},\mathbb{R})$ be defined as $f(x) = \frac{2}{\pi}\operatorname{arctg}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)$.

- Study the graph of f.
- Let $\dot{x} = f(x)$. Draw the phase-portrait and discuss quality of equilibria.
- Let $\ddot{x} = -f'(x)$. Draw the phase-portrait and discuss quality of equilibria.
- For $\ddot{x} = -f'(x)$. Establish the subset $\mathcal{E}$ of $\mathbb{R}$ of energy values corresponding to periodic orbits.
- For $\ddot{x} = -f'(x)$. Let $E = \frac{1}{\pi}\operatorname{arctg}(1/6)$. Is every orbit of energy E periodic? Write the formula (without solving integral) for T.

Ex 2 Let $V(x) = -1/x - 4x + x^3$. $(x>0)$

- Write Newton eqs for this conservative system and the corresponding v.f.
- Determine eq. and stability with spectral method.
- Determine a first integral and draw the phase-portrait.

Ex 3 Let $\ddot{x} = -\omega^2 \sin x + \Omega^2 \sin x \cos x$

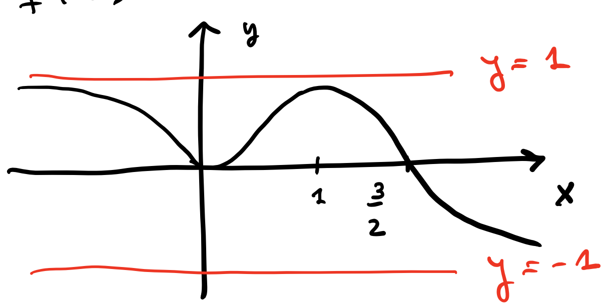
- Determine and classify eq. of the linearized system. Draw the corresponding bifurcation diagram.
- Determine (un)stability of eq. of the original system (by phase-portrait).
- Let $\ddot{x} = -\omega^2 \sin x + \Omega^2 \sin x \cos x - 2\mu\dot{x}$, $\mu>0$. Determine the quality of eq. $(0,0)$ by an appropriate Lyap. function.

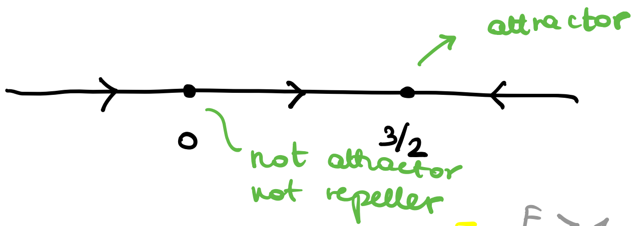
$\boxed{1}$ $f(x) = \frac{2}{\pi}\operatorname{arctg}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)$

$$\lim_{x\to\pm\infty} f(x) = \frac{2}{\pi}\cdot\left(\mp\frac{\pi}{2}\right) = \mp 1$$

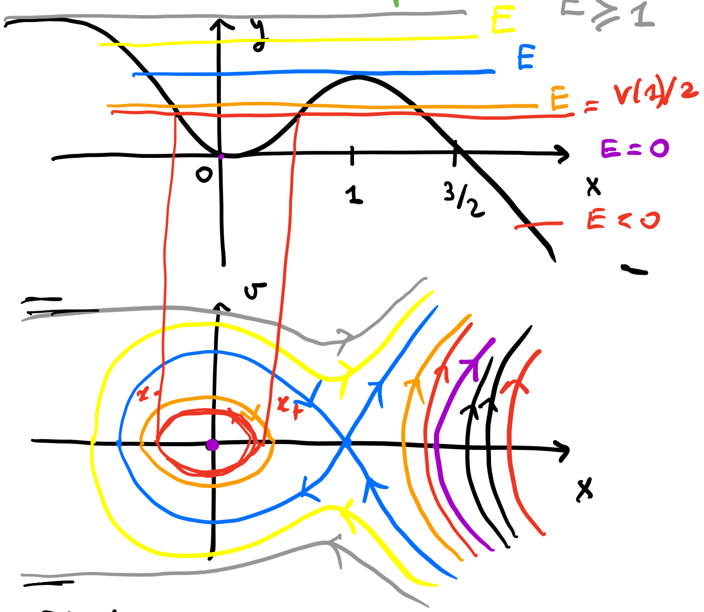
$$f'(x) = \frac{2}{\pi}\cdot\underbrace{\frac{1}{1+\left(\frac{x^2}{2}-\frac{x^3}{3}\right)^2}}_{>0}\underbrace{\left(x-x^2\right)}_{=x(1-x)} = 0 \iff \begin{matrix} x=0 \\ x=1 \end{matrix}$$

$f(0) = 0$





$\dot{x} = f(x)$



$E_1 = (0,0)$

$E_2 = (1,0)$

$\underline{E \geq 1}$ $$\frac{1}{2}v^2 + \frac{2}{\pi}\arctan\left(\frac{x^2}{2} - \frac{x^3}{3}\right) = E$$

Then $$\lim_{x\to+\infty} \frac{1}{2}v^2 = \lim_{x\to+\infty} E - \overbrace{\frac{2}{\pi}\arctan\left(\frac{x^2}{2} - \frac{x^3}{3}\right)}^{-1} = E+1$$

$$v \to \pm\sqrt{2(E+1)}$$

$$\lim_{x\to-\infty} \frac{1}{2}v^2 = E-1 \quad , \quad v \to \pm\sqrt{2(E-1)}$$

$$\mathcal{E} = [0, V(1)] = \left[0, \frac{2}{\pi}\arctan\left(\frac{1}{6}\right)\right]$$

(fixed points are periodic orbits!)

$$E = \frac{1}{\pi}\arctan\left(\frac{1}{6}\right) = \frac{V(1)}{2} \in \mathcal{E}$$

At this level, we have a periodic orbit.

$$T = 2\int_{x_-}^{x_+} \frac{1}{\sqrt{2(E - V(x))}}\, dx$$

—x—x—

$\boxed{2}$ For this conservative v.f.

$$\ddot{x} = -V'(x) = -\frac{1}{x^2} + 4 - 3x^2$$

$$\left(V(x) = -\frac{1}{x} - 4x + x^3\right)$$

$$X(x,v) = \begin{pmatrix} v \\ -1/x^2 + 4 - 3x^2 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix}$$

Equilibrium configurations?

$$-\frac{1}{x^2} + 4 - 3x^2 = 0 \iff -1 + 4\overbrace{x^2}^{t} - 3\overbrace{x^4}^{t^2} = 0$$

$$\iff x = \pm\sqrt{\frac{4 \pm \sqrt{4}}{6}} = \pm\sqrt{\frac{4 \pm 2}{6}} = \begin{cases} 1 \\ \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} \end{cases}$$

$$\underline{\underline{x > 0}}$$

EQUILIBRIA

$(1, 0)$ AND $\left(\frac{1}{\sqrt{3}}, 0\right)$

Stability ?! → Draw the $V(x)$

OR

→ Try with spectral method.

$$JX(x,v) = \begin{pmatrix} 0 & 1 \\ \frac{2}{x^3} - 6x & 0 \end{pmatrix}$$

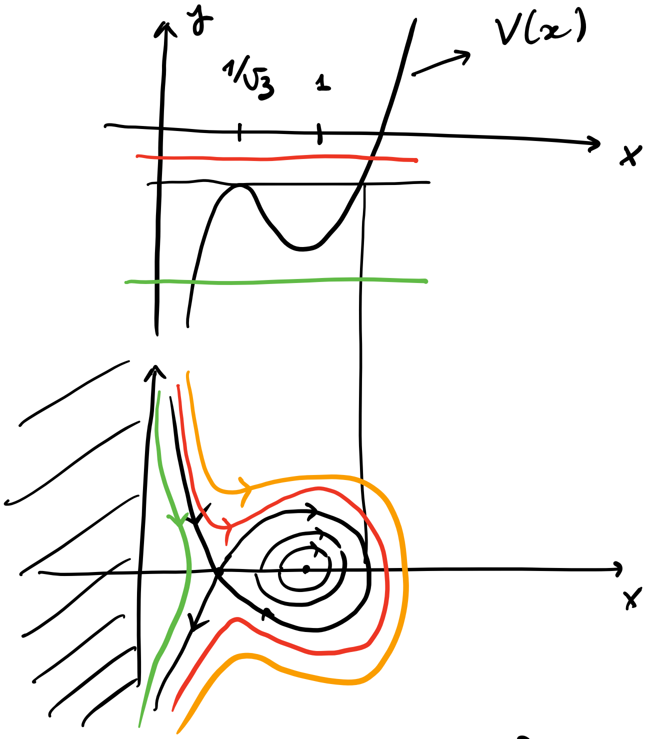
$$JX(1,0) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \to \lambda_{1,2} \text{ purely imag.}$$

$\Rightarrow$ we cannot conclude!

$$JX\left(\frac{1}{\sqrt{3}}, 0\right) = \begin{pmatrix} 0 & 1 \\ 4\sqrt{3} & 0 \end{pmatrix} \to \lambda_{1,2} \text{ real } \lessgtr 0 \text{ eigenv.}$$

UNSTABLE for the original (non linear) syst.

$E = \frac{1}{2}v^2 + V(x)$ is a first integral.



3 $\ddot{x} = -\omega^2 \sin x + \Omega^2 \sin x \cos x$

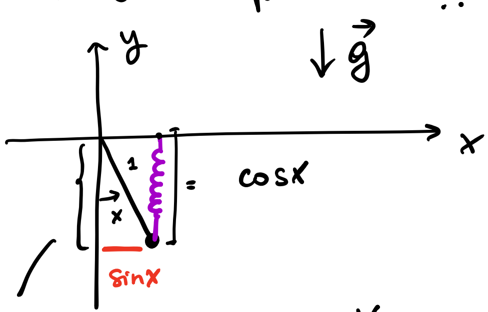
$$\begin{cases} \dot{x} = v \\ \dot{v} = -V'(x) = -\omega^2 \sin x + \Omega^2 \sin x \cos x \end{cases}$$

$V(x) = \boxed{-\omega^2 \cos x} + \boxed{\frac{1}{2}\Omega^2 \cos^2 x}$

AS A GRAV. POTENTIAL

AS AN ELASTIC POTENTIAL

Mechanical interpretation ?!



$V_{gr} = mg\,y_P$

$V_{el} = \frac{1}{2}K(\text{length spring})^2$

$-\cos x$

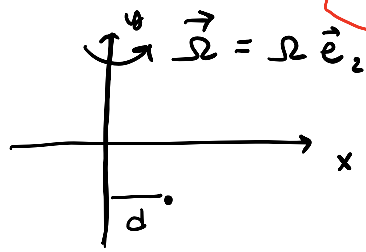
$V_{gr} = -\omega^2 \cos x$

$V_{el} = \frac{1}{2}\underbrace{\Omega^2}_{=K} \cos^2 x$

<u>Alternative potential:</u>

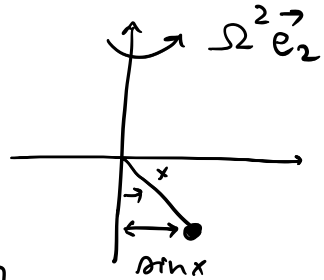
$$V(x) = -\omega^2\cos x - \frac{1}{2}\Omega^2\sin^2 x$$

AS A CENTRIFUGAL POTENTIAL.



$$V_{cf} = -\frac{1}{2}\Omega d^2$$

In such a case



$$\Rightarrow \quad V_{cf} = -\frac{1}{2}\Omega^2\sin^2 x$$

$\boxed{\text{LINEARIZED SYSTEM}}$

EQUILIBRIA

$$V'(x) = 0 = -\omega^2\sin x + \Omega^2\sin x\cos x$$

$$= \sin x\big(\underbrace{-\omega^2 + \Omega^2\cos x}\big) = 0$$

if $x = 0$ OR $x = \pi$ OR $\cancel{\Omega^2}\cos x = \dfrac{\omega^2}{\Omega^2} \Leftrightarrow$

$x = \pm\arccos\left(\dfrac{\omega^2}{\Omega^2}\right)$ only when $\dfrac{\omega^2}{\Omega^2} < 1$

($= 1$ correspond to $x = 0$)

$$J(x,v) = \begin{pmatrix} 0 & 1 \\ -V''(x) & 0 \end{pmatrix}$$

$$-V''(x) = -\omega^2\cos x + \Omega^2\cos^2 x - \Omega^2\sin^2 x$$

$\boxed{\text{I}}$ $x = 0$ $\rightarrow$ $(0,0)$ EQUILIBRIUM

$$-V''(0) = -\omega^2 + \Omega^2$$

trace $J(0,0) = 0$

det $J(0,0) = V''(0)$

when det $> 0 \iff \omega^2 - \Omega^2 > 0 \iff \Omega^2 < \omega^2$ $\searrow \Omega^2/\omega^2 < 1$
(center) $\to$ STABILITY
when det $< 0 \iff \Omega^2 > \omega^2$ (saddle) (UNSTABILITY)

$\boxed{2}$ $x = \pi$ $\to$ $(\pi, 0)$ EQUILIBRIUM

$-V''(\pi) = \omega^2 + \Omega^2 \Rightarrow$ det < 0 ALWAYS.

$\Rightarrow$ SADDLE $\to$ UNSTABLE.

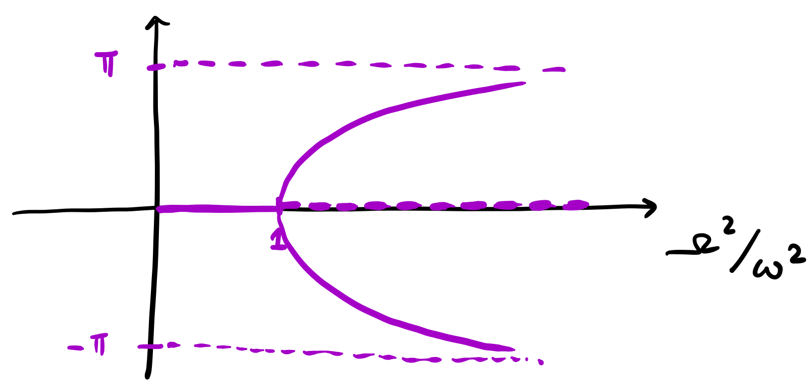
$\boxed{3}$ Suppose now $\Omega^2 > \omega^2 \Rightarrow$ we have also the other 2 equilibria $\left(\pm \arccos\left(\omega^2/\Omega^2\right), 0\right)$

$V''\left(\pm\arccos\left(\omega^2/\Omega^2\right)\right) > 0 \Rightarrow$ det > 0
trace $= 0$

$\Rightarrow \left(\pm x^*, 0\right)$ are centers. $\Rightarrow$ STABLE!

$x^* = \arccos\left(\omega^2/\Omega^2\right)$

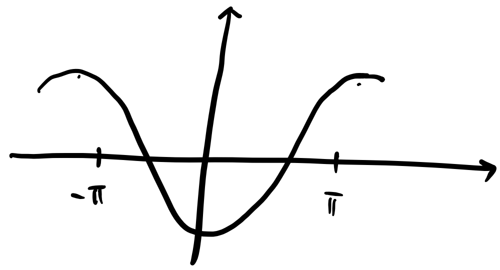
Bifurcation diagram for the linearized system

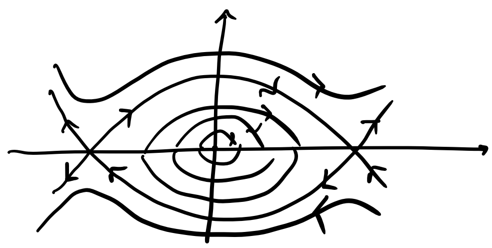


$\boxed{\text{NON-LINEAR ORIGINAL SYSTEM}}$

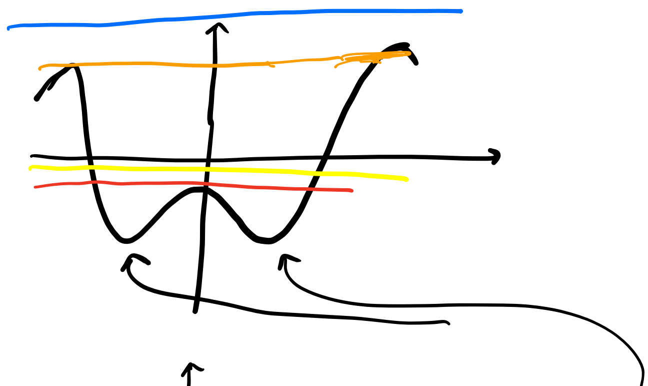
$$V(x) = -\omega^2 \cos x + \frac{1}{2}\Omega^2 \cos^2 x$$

If $\Omega^2/\omega^2 \leq 1$

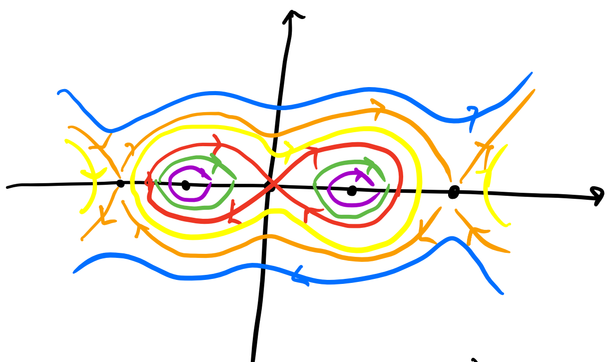




If $\Omega^2/\omega^2 > 1$



Double "hole" of potential



$X(x,v)$

With dissipation

$$\begin{cases} \dot{x} = v \\ \dot{v} = -V'(x) - 2\mu v \end{cases}$$

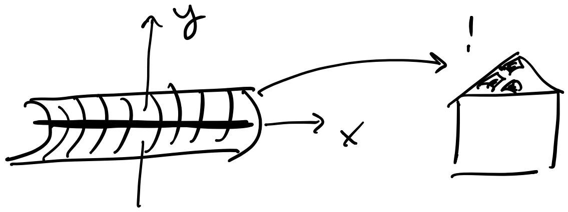
$$J(x,v) = \begin{pmatrix} 0 & 1 \\ -V''(x) & -2\mu \end{pmatrix}$$

$E(x,v) = \frac{1}{2} v^2 + V(x)$ AS LYAPUNOV FUNCTION

$L_X E(x,v) = \nabla E(x,v) \cdot X(x,v) =$

$$= \left(V'(x),\ v \right) \cdot \begin{pmatrix} v \\ -V'(x) - 2\mu v \end{pmatrix} =$$

$$= \cancel{V'(x) v} - \cancel{V'(x) v} - 2\mu v^2 = -2\mu v^2 \leq 0$$



We can conclude <u>only</u> stability.