The uncertain path of *determinism* in Classical Mechanics

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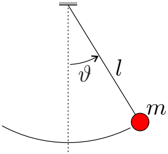
Course of Mathematical Physics, [redacted]



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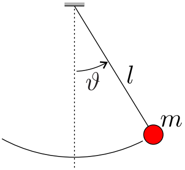

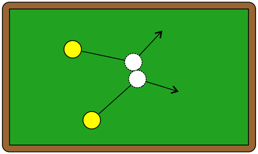

The kingdom of the deterministic approach

Determinism

– The laws of Mechanics are such that the present state of a system completely determines the future (and the past) of the system

state : a suitable set of variables (e.g.: position and velocity of all particles); “initial datum”

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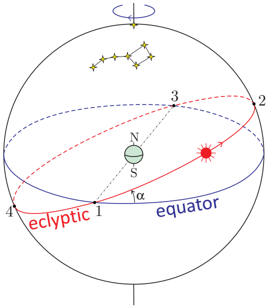
Intrinsically random system – *aleatoric*

Intrinsic unpredictability
(atomistic philisophers, V sec. b.C.)

Possibly: probabilistic laws
statistic regularity

alea = dice



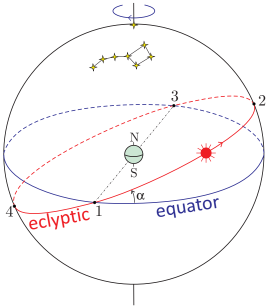
Determinism: the “two-spheres model”

Two angles: $\varphi \mapsto \varphi + \omega t$ (day)

$\Phi \mapsto \Phi + \Omega t$ (year)

It explains:

- equinoxes, solstices; seasons
- diff. length of day and night, along the year
- alternation of constellations in the night sky



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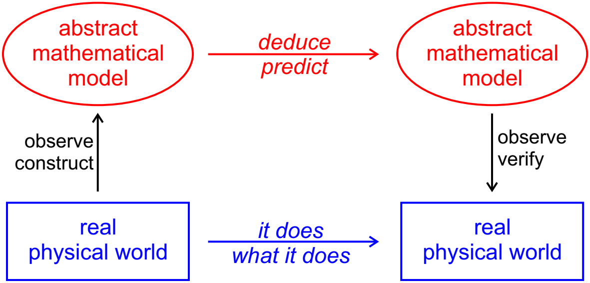
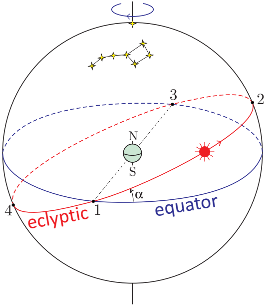
“Model”:



Not really satisfactory



Does not explain:

- exact duration of seasons
- motion of planets
- ...

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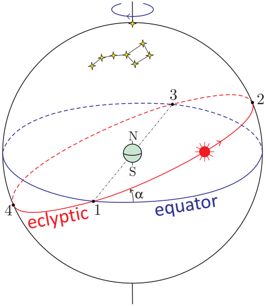

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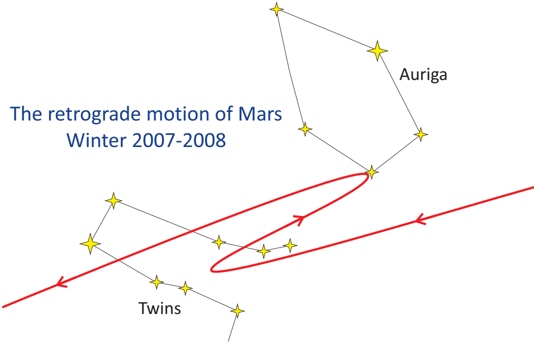
Planets :

complicated motions

not uniform

The retrograde motion of Mars
Winter 2007-2008



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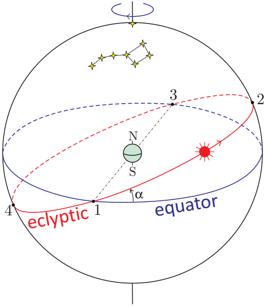

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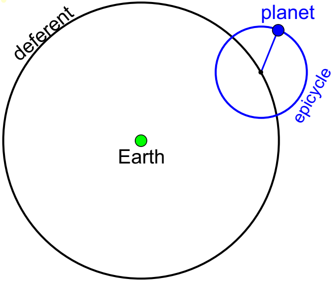
More complicated model :

Ptolemy



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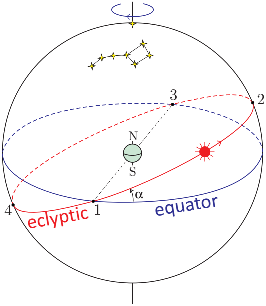

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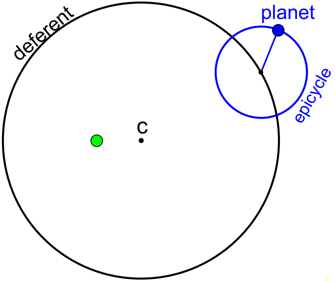

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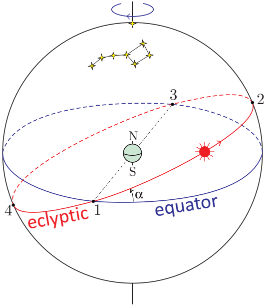

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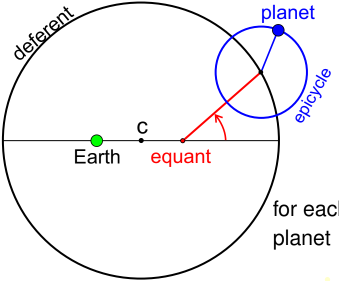

for each planet

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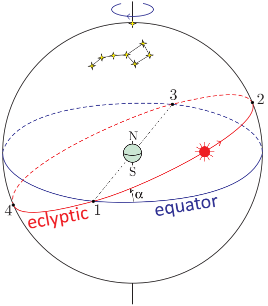

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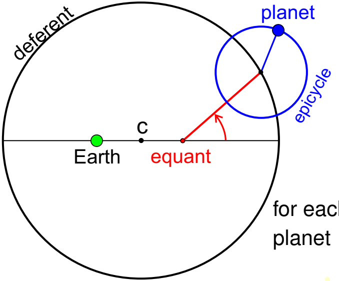

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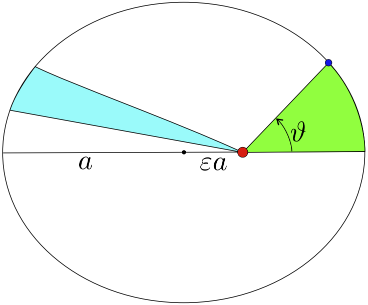
More complicated model :

Ptolemy

Copernicus simplifies ,
not really better

Losing models,
no hope this way !

Towards a better model



Kepler laws: motions on ellipses

low of areas

$T^2 = K a^3$

The scientific revolution (Newton, *Principia* 1687; after Kepler, Galileo...)

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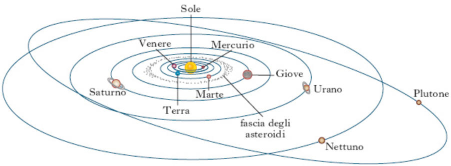

1) $m\vec{a} = \vec{F}$

2) $F = G\dfrac{M\,m}{r^2}$, attractive

+ differential calculus

$$\frac{\mathrm{d}^2\vec{r}_i}{\mathrm{d}t^2} = \frac{1}{m_i}\vec{F}_i(\vec{r}_1,\ldots,\vec{r}_n)$$

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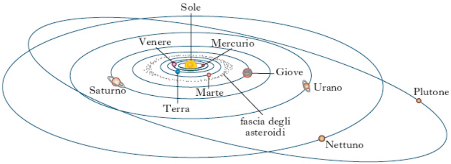

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$\Longleftrightarrow$ Kepler laws

(+ terrestrial physics...)

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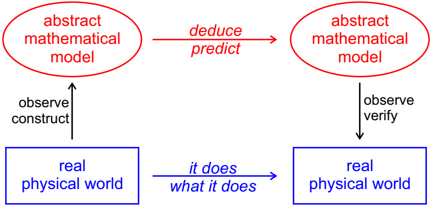

The winning paradigm !

– developments, generalizations; *mathematical rigor*

Euler

Lagrange

Laplace...

} "Analitical Mechanics" – XVIII sec

... Cauchy ...

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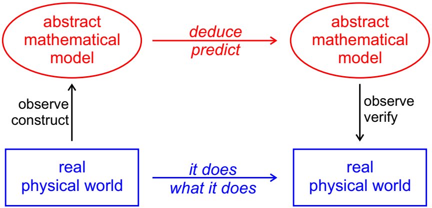
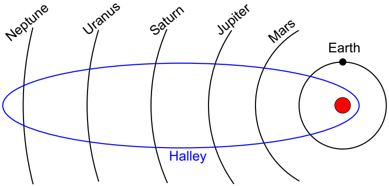


in the abstract model
determinism becomes
a theorem

(Cauchy, beginning of XIX century)

An *impressive* progression of confirmations

- it includes new phenomena: e.g., the flattened orbits of comets (Halley in 1682 predicts the return of his comet in 1757; will be in 1758)



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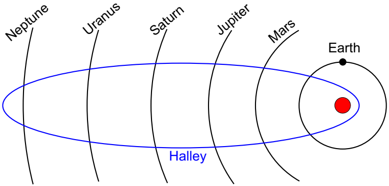

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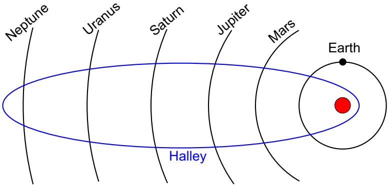

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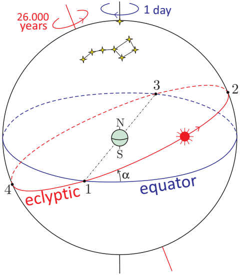

Hypparcus (II century b.C.)

↓

Newton, *Principia*, 1687

(Euler, d’Alembert)

$T \simeq 26.000$ years

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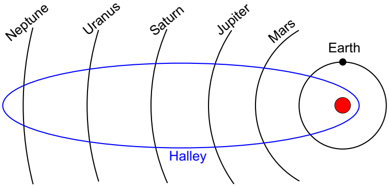

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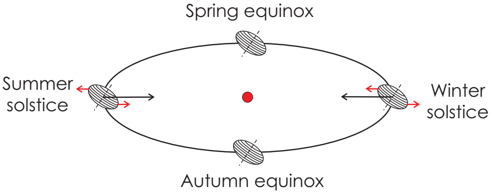

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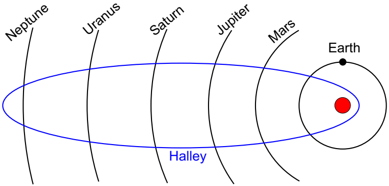

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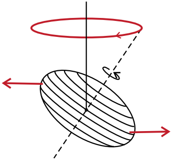

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Laplace : new theoretical methods, corrections to elliptic motions

much more accurate observations; statistical methods

computation and observation do correspond !

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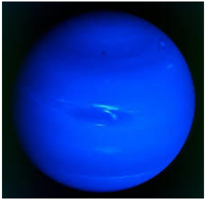
- The discovery of Neptune:



Neptune

1781 : Herschel discovers Uranus; $T = 84$ years

1821 on: its motion has little unexplained irregularities

Corrections to Newton's law (to the model) ?

Unknown celestial body ?

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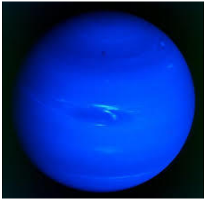

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Adams, Le Verrier, 1846 : compute the position

Galle, 1846, finds the planet where predicted (diff. 1°)

absolute confidence in the model

25 september 1846, Galle to Le Verrier:

... the planet, whose position you predicted, really exist. The same day I received your letter, I found a star of 8^{th} *magnitude, which did not appear in the collection of the sky maps published by the Berlin Royal Academy. The observation of the next night decided it was the planet you were searching for.*

The reply:

Thanks to you, we are definitely in possess of the new world. (...) The Bureau des Longitudes *chose the name* Neptune*. The sign: a trident.*

Laplace, *Essai philosophique sur les probabilités*, 1812 :

“ We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes. ”

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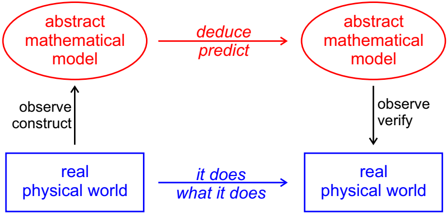


Luminous deterministic view

Beyond the mechanic–deterministic view
the crisis of mechanics between XIX and XX centuries

– not only mechanics : elettromagnetism
  Independent. Which is fundamental ?

Beyond the mechanic–deterministic view

the crisis of mechanics between XIX and XX centuries

– not only mechanics : elettromagnetism

Independent. Which is fundamental ?

– end XIX – beginning XX sec :

general rethinking inside mechanics

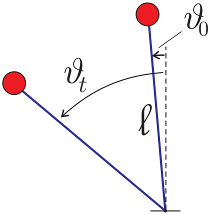
Predictability ?

$\rightarrow$ modern notion of chaotic motion

Henry Poincaré (1854 – 1912) :

sensitive dependence on initial data

My favourite example



$$\ell = 10\,\mathrm{cm}$$

$$\frac{\mathrm{d}^2\vartheta}{\mathrm{d}t^2} = +\,\omega^2 \sin\vartheta$$

$$\simeq \omega^2\,\vartheta\,; \qquad \omega = \sqrt{\frac{g}{\ell}} \simeq 10\ \mathrm{sec}^{-1}$$

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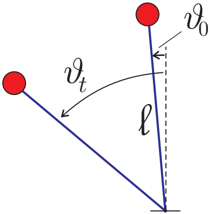

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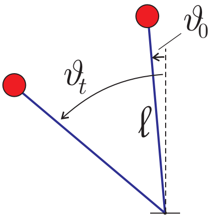
$$\simeq \omega^2\,\vartheta\ ; \qquad \omega = \sqrt{\frac{g}{\ell}} \simeq 10\ \mathrm{sec}^{-1}$$

Solution, for $v_0 = 0$:

$$\vartheta_t = \tfrac{1}{2}\,\vartheta_0 \left(e^{\omega t} + e^{-\omega t}\right) \simeq \tfrac{1}{2}\,\vartheta_0\, e^{\omega t}$$

How much time to reach ϑ_t ?

$$t \simeq \frac{1}{\omega}\,\log\frac{2\vartheta_t}{\vartheta_0}$$



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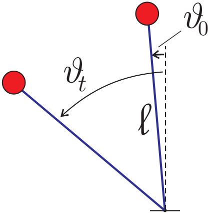
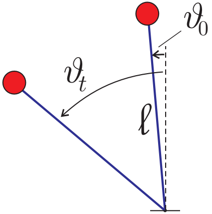


$$t = \frac{1}{\omega} \log \frac{2\vartheta_t}{\vartheta_0} \quad = \frac{1}{\omega} \log \frac{1}{\vartheta_0}$$

$$\vartheta_t = 0.5\,\text{rad}$$



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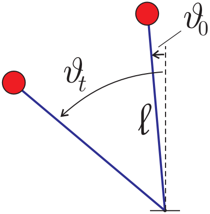
$\vartheta_0 = 10^{-3}$ rad $\qquad t = 0.7$ sec



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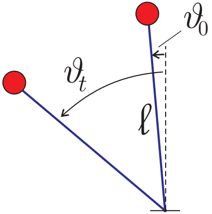
10^{-6}	1.4



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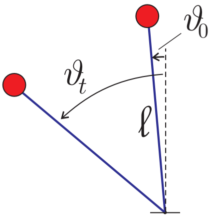
10^{-12}	2.8



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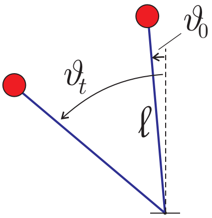

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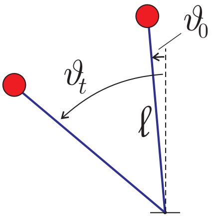
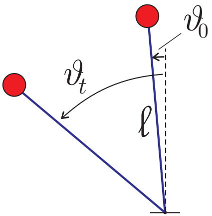
10^{-6}	1.4	attraction of the Moon
10^{-12}	2.8	
10^{-24}	5.6	



$$t = \frac{1}{\omega} \log \frac{2\vartheta_t}{\vartheta_0} = \frac{1}{\omega} \log \frac{1}{\vartheta_0}$$

$\vartheta_t = 0.5\,\mathrm{rad}$

$\vartheta_0 = 10^{-3}$ rad	$t = 0.7$ sec	
10^{-6}	1.4	attraction of the Moon
10^{-12}	2.8	a person, 1m distance
10^{-24}	5.6	



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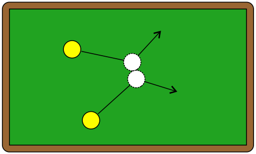
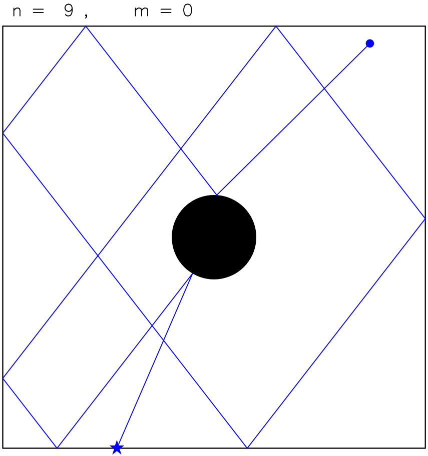
10^{-12}	2.8	a person, 1m distance
10^{-24}	5.6	a mosquito, 1km distance

sensitive dependence on initial data

A simplified billiard



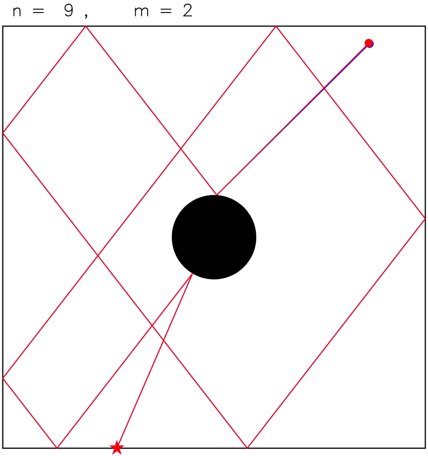


A simplified billiard

second trajectory,

$\Delta\vartheta = 10^{-6}$ rad

9 collisions

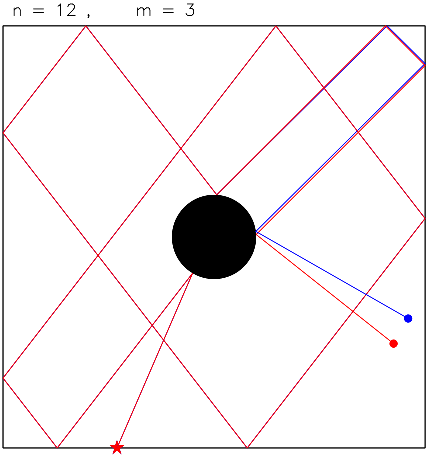


A simplified billiard

second trajectory,

$\Delta\vartheta = 10^{-6}$ rad

9 collisions
12 collisions

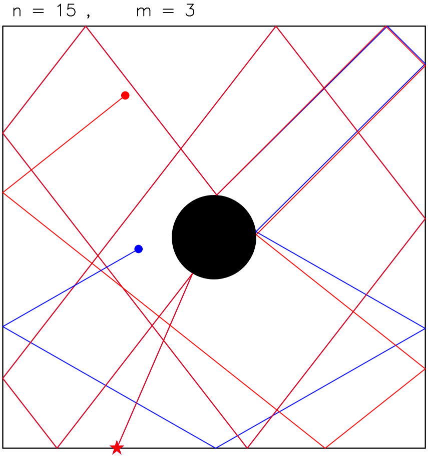


A simplified billiard

second trajectory,

$\Delta\vartheta = 10^{-6}$ rad

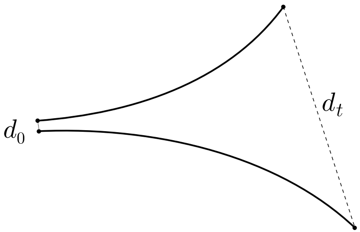
9 collisions
12 collisions
15 collisions



sensitive dependende on the initial datum

for generic initial data !

“*Chaotic*” system :

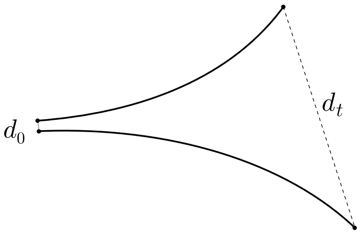


$$d_t \simeq d_0\, e^{\lambda t}\,, \qquad \lambda > 0$$

exponential instability

for a relevant set of motions
(positive measure)

"Chaotic" system :



$$d_t \simeq d_0 \, e^{\lambda t} , \qquad \lambda > 0$$

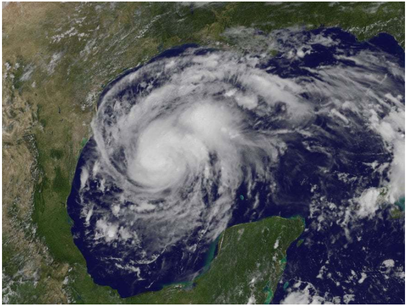
exponential instability

for a relevant set of motions
(positive measure)

little pendulum: $\lambda = \omega = 10 \, \mathrm{sec}^{-1}$; $\qquad e^{10 \times 5.6} \simeq 10^{24}$

The “butterfly effect”

Morpho-Menelaus



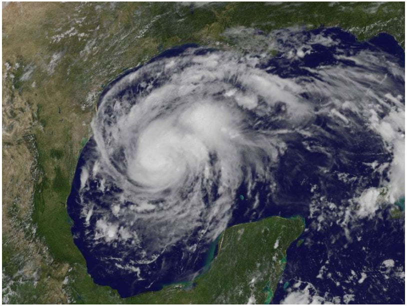
the Hurricane *Harvey* (Texas, 2017)

Eduard N. Lorenz (1968; a talk, 1972):

“Can the flap of a butterfly’s wings in Brazil set off a tornado in Texas?”

The “butterfly effect”

Morpho-Menelaus



the Hurricane *Harvey* (Texas, 2017)

Eduard N. Lorenz (1968; a talk, 1972):

“Can the flap of a butterfly’s wings in Brazil set off a tornado in Texas?”

In turbulence conditions: yes, it might happen (time scale: a few weeks)

In the Solar System ?

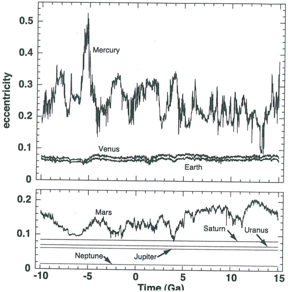
– Mercury: quite pronunced chaotic motions ,
  $\simeq$ 10 millions years

– Venus, Earth: small chaotic oscillations

– Mars: intermediate behavior

– outer planets: thin chaotic regions ?

(1990 – today; J. Laskar)



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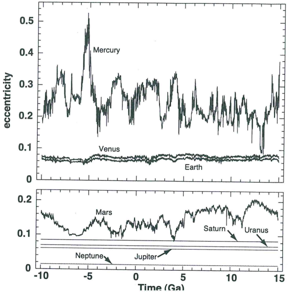
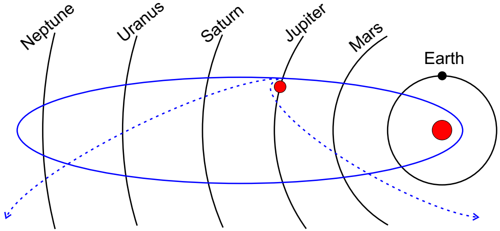


- comets:

  similar to billiards,
  a few thousands of years

  (Lexell, 1770)

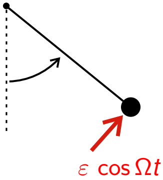


infiltrations of chaos

The *forced pendulum*

add a small external periodic force:

$$\frac{d^2\vartheta}{dt^2} = -\omega^2 \sin\vartheta + \varepsilon \cos \Omega t$$



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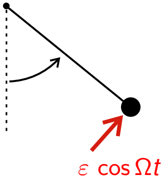
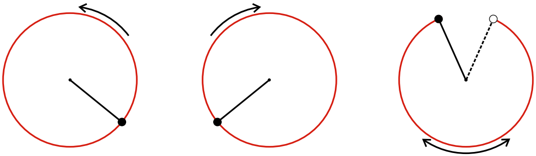


no force, $\varepsilon = 0$:



LLLL... *RRRR...* *LRLR...*

The *forced pendulum*

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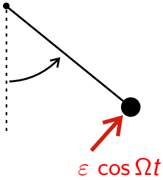

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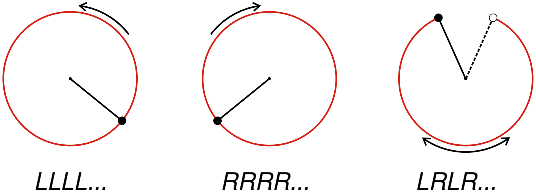
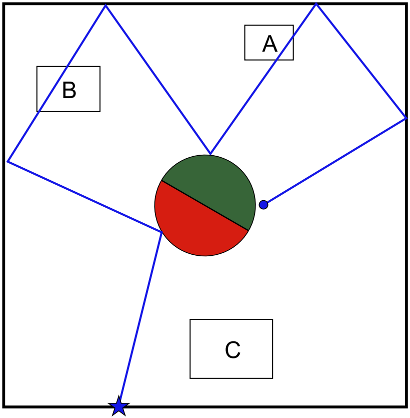


Theorem : for small $\varepsilon > 0$ all sequeces are realized

Unpredictability $\longrightarrow$ probability ?

Chaotic motions, very irregular, may become statistically regular *("ergodic problem", difficult)*



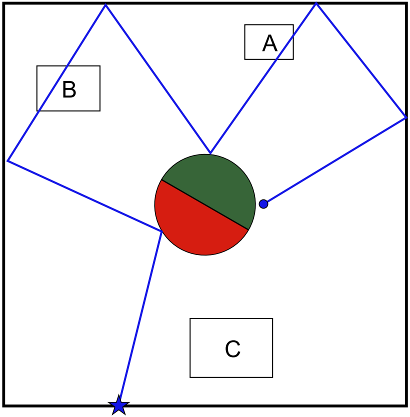
Chaotic billiards :

a generic trajectory obeys "simple" statistical rules

(Ya. Sinai, 1962)

Unpredictability $\longrightarrow$ probability ?

Chaotic motions, very irregular, may become statistically regular *(“ergodic problem”, difficult)*



probability
to be in A, B, C...
proportional to the area

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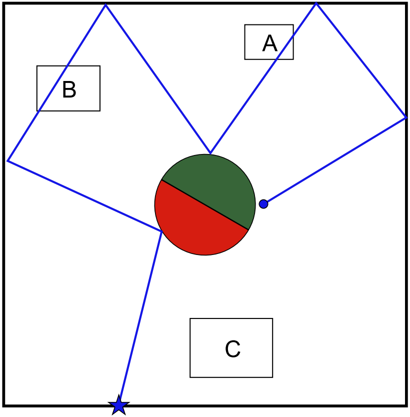

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probability of a collision
proportional
to the lenght of the border

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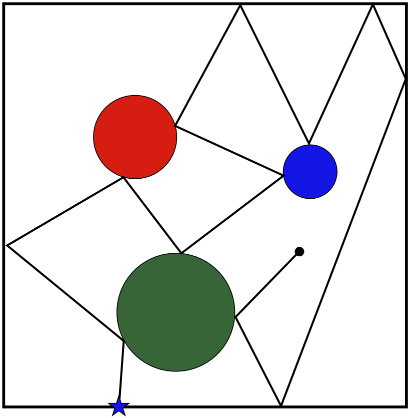

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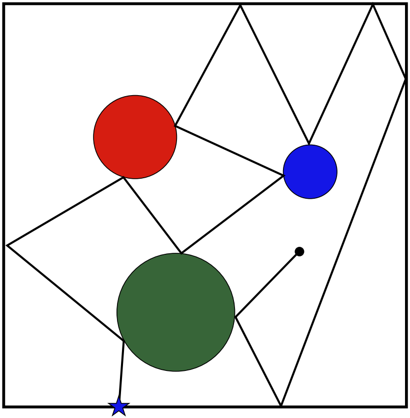


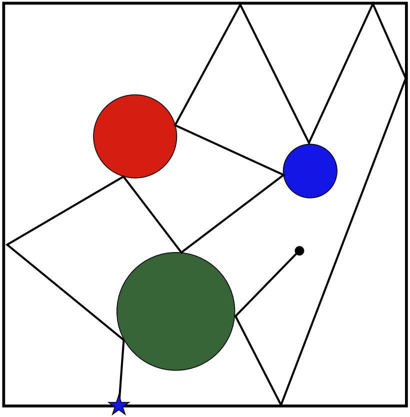
Unpredictability of trajectories

$\downarrow$

statistical predictability

Unpredictability $\longrightarrow$ probability ?

Chaotic motions, very irregular, may become statistically regular *("ergodic problem", difficult)*



deterministic dynamics

$\downarrow$

precise statistical laws

How computers do produce random numbers ?

(criptography...)

Computers:

absolutely deterministic devices

How computers do produce random numbers ?

(criptography...)

Computers:

absolutely deterministic devices

A chaotic dynamics is used

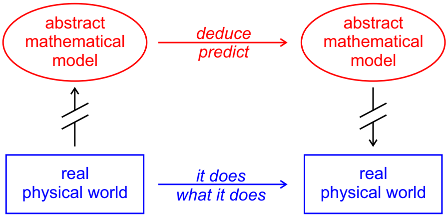
(sophisticated)

Conclusion ?

The instab. of chaotic motions
makes uncertain
the construction of a model

deterministic model,
  trajectories ?

probabilistic model,
  statistical laws ?



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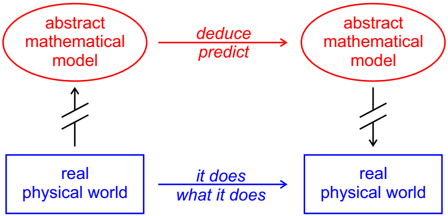

determinism and probability are complementary descriptions

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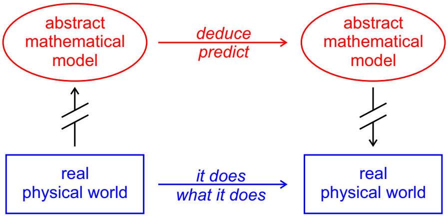

determinism and probability are complementary descriptions

the pure deterministic image can be misleading

true, but misleadind

Only by chance ?

" The greatest chance is the birth of a great man. It is only by chance that the meeting occurs of two genital cells of different sex that contain precisely, each on its side, the mysterious elements whose mutual reaction is destined to produce genius. (...)

How little it would have taken to make the spermatozoid which carried them deviate from its course. It would have been enough to deflect it a hundredth part of an inch, and Napoleon would not have been born and the destinies of a continent would have been changed. No example can give a better comprehension of the true character of chance. "

Henry Poincaré, *Le hazard*, 1907

Also in:	*Science et Méthode*, IV — *Le hazard*
English:	*Science and Method*, IV – *Chance*
Itanian:	*Scienza e metodo*, IV – *Il caso*