Proposition: $f : D \longrightarrow \mathbb{R}$ $\quad g : D \longrightarrow \mathbb{R}$

f and g continuous at $\xi \in Acc(D) \cap D$

- $f+g : x \longmapsto f(x)+g(x)$ is continuous at ξ
- $f \cdot g \longmapsto f(x) \cdot g(x)$ is continuous at ξ
- if $g(\xi) \neq 0$

  $\frac{f}{g}$ is continuous at ξ.

Proof:

$$\lim_{x \to \xi} f(x) \cdot g(x) \overset{?}{=} (f \cdot g)(\xi) = f(\xi) \cdot g(\xi)$$

$$\parallel$$

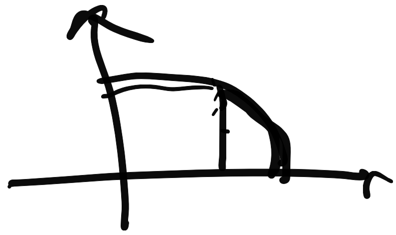
$$\lim_{x \to \xi} f(x) \cdot \lim_{x \to \xi} g(x) = f(\xi) \cdot g(\xi)$$

Same trivial proof for sum and product. q.e.d.

Exercise: sin is continuous at every point $\xi \in \mathbb{R}$

Let's take $\xi = 0$

$$\boxed{\lim_{x \to 0} \sin x \overset{?}{=} 0}$$



We know $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$

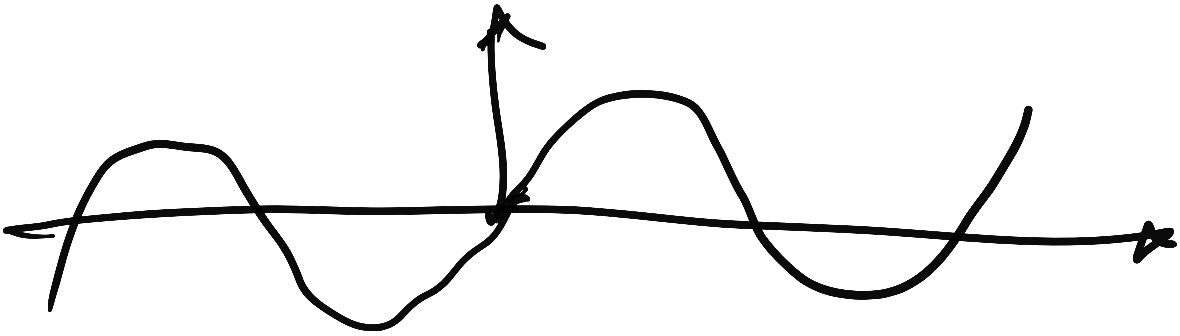
$$\Updownarrow$$

$$\sin x = x + o(x) \qquad x \to 0$$

$$\lim_{x \to 0} \sin x = \lim_{x \to 0} x + \lim_{x \to 0} o(x) = 0 + 0 = 0 = \sin 0$$

(we have used $\displaystyle\lim_{x \to 0} o(x) = 0$ $\qquad \displaystyle\lim_{x \to 0} o(x) = \lim_{x \to 0} \underbrace{\frac{o(x)}{x}}_{0} \cdot \underbrace{x}_{0} = 0$)

so sin is continuous at 0



$\xi \in \mathbb{R}$

$$\lim_{x\to\xi} \sin x \overset{?}{=} \sin \xi$$

$$\boxed{\lim_{x\to\xi}\sin x} \underset{x=\xi+y}{=} \lim_{y\to 0}\sin(\xi+y) =$$

$$\lim_{y\to 0}\Big(\underbrace{\sin\xi\cos y}_{\sin\xi} + \underbrace{\cos\xi\sin y}_{0}\Big) = \boxed{\sin\xi}$$

Let us prove cos is continuous at zero: $\lim_{y\to 0}\cos y = \cos(0) = 1$

$$\lim_{y\to 0}\frac{1-\cos y}{y^2} = \frac{1}{2} \iff \cos y = 1 - \frac{1}{2}y^2 + o(y^2)$$

$$\lim_{y\to 0}\cos y = 1 + \lim_{y\to 0}\left(-\frac{1}{2}y^2 + o(y^2)\right)$$

$$= 1 + 0 = 1$$

So: $\sin$ is continuous at every $\xi \in \mathbb{R}$

$\cos$ is continuous at every $\xi \in \mathbb{R}$

Indeed:

$$\boxed{\lim_{x \to \xi} \cos(x)} \underset{x = y + \xi}{=} \lim_{y \to 0} \cos(\xi + y) =$$

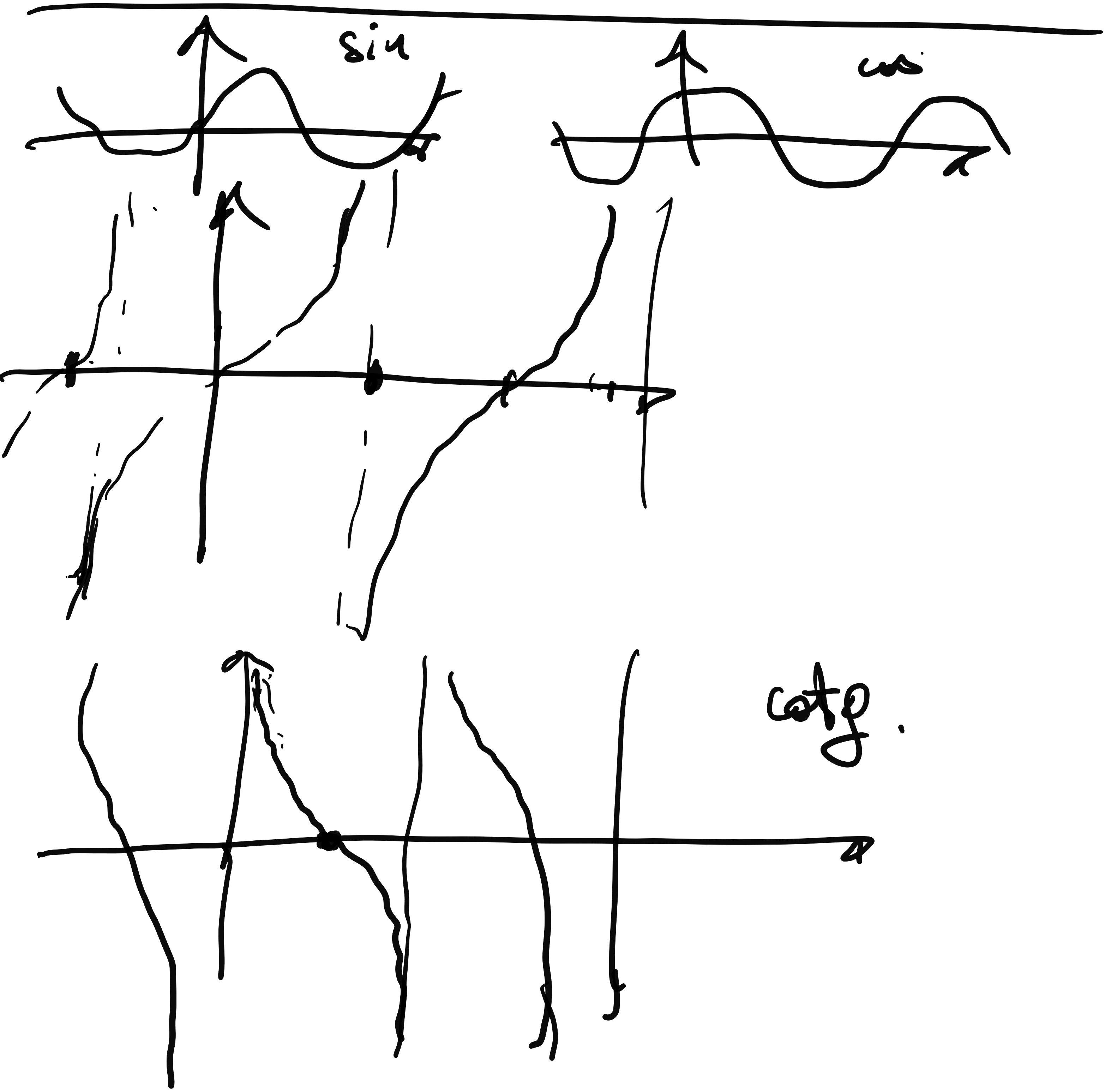
$$\lim_{y \to 0} \big(\underbrace{\cos\xi \cos y}_{\to \cos\xi} - \underbrace{\sin\xi \sin y}_{\to 0} \big) = \boxed{\cos\xi}$$

<u>Question</u>: is $\boxed{tg}$ continuous at every point ξ of its domain

$$D = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi ,\ k \in \mathbb{Z} \right\}$$

$\xi \in D$ $\qquad tg\, x = \dfrac{\sin x}{\cos x}$ is continuous at ξ. We have $\cos\xi \neq 0$ apply proposition saying that quotient of continuous is continuous

Similarly, $\cot g = \frac{\cos x}{\sin x}$ is continuous at every $\xi \in D = \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$



$f : D \longrightarrow \mathbb{R}$

$f \in \mathcal{C}(D)$: "f is continuous at every $\xi \in D$"

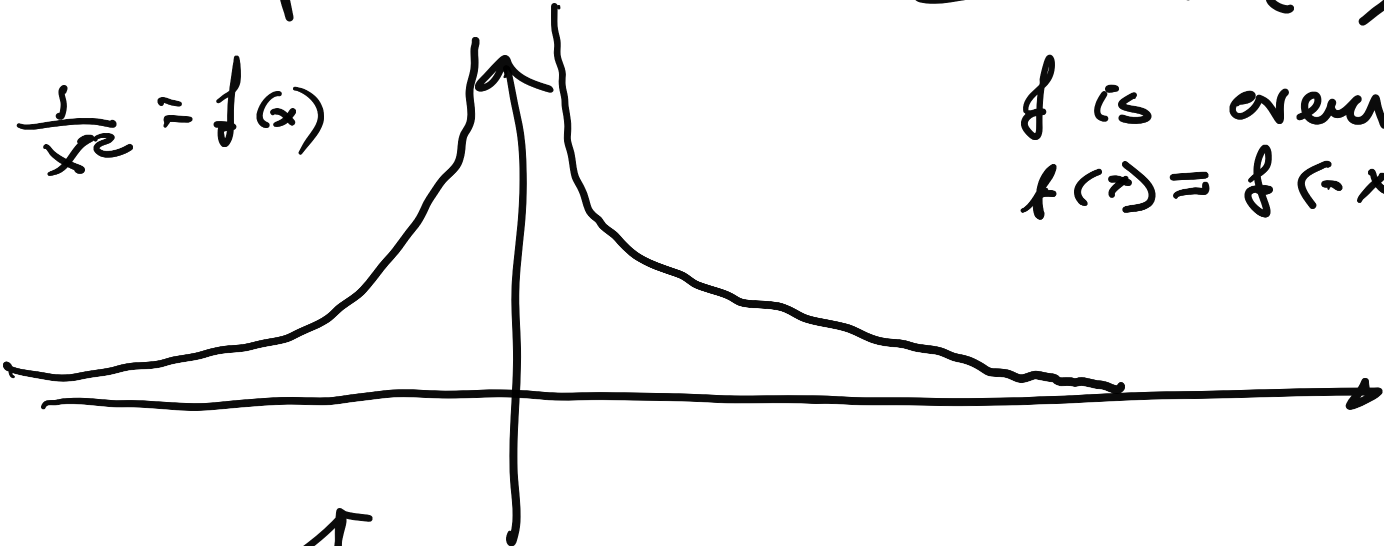
Example

$$\frac{1}{x^n} \qquad n \in \mathbb{N}$$

is continuous at every point of its domain $D = \mathbb{R} \setminus \{0\}$

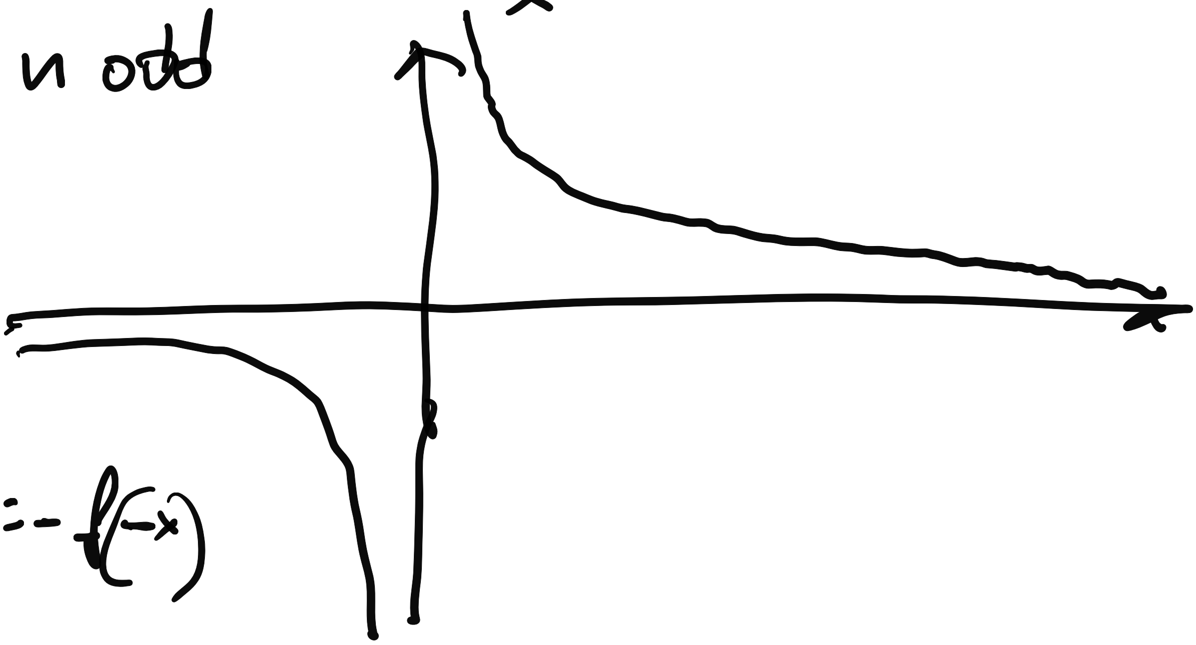
$\frac{1}{x^2} = f(x)$

f is even
$f(x) = f(-x) \ \forall x$



same form $\frac{1}{x^n}$ n even

$\frac{1}{x^n}$, n odd
$f(x)$



$f(x) = -f(-x)$

$$f(x) = \frac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_m x^m}$$

is continuous at every point of the Domain $D=\left\{x\in\mathbb{R} \text{ s.t. } b_0+b_1x+\dots+b_mx^m\neq 0\right\}$

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is $f(x)=\cos(\sin(x))$ continuous at $\xi\in\mathbb{R}$?

is
$$g(x)=\frac{\left(tg(x)\right)^3+\cos(\pi x)}{\cos\left(\frac{x^2+e}{2}\right)+\sin\left(x^3\right)}$$
continuous at every point of its domain?

$$f(x)=g\left(h(x)\right)=g\circ h(x)$$

where $g(y)=\cos y$

$h(x)=\sin x$

Theorem $h: D \longrightarrow \mathbb{R}$

$g: E \longrightarrow \mathbb{R}$ $\quad h(D) \subseteq E$

$f(x) = g \circ h(x) = g(h(x))$

$\xi \in Acc(D) \cap D$ suppose that

h is continuous at ξ

g is continuous at $h(\xi)$

Then $f = g \circ h$ is continuous at ξ.

Recall change of variable theorem: $f(x) = g(h(x))$

i) $\lim_{x \to \xi} h(x) = y_0$

ii) $\lim_{y \to y_0} g(y) = \boxed{l}$

iii) $h(x) \neq y_0$ in a neigh. I of ξ

then $\lim_{x\to\xi} f(x) = \boxed{l}$

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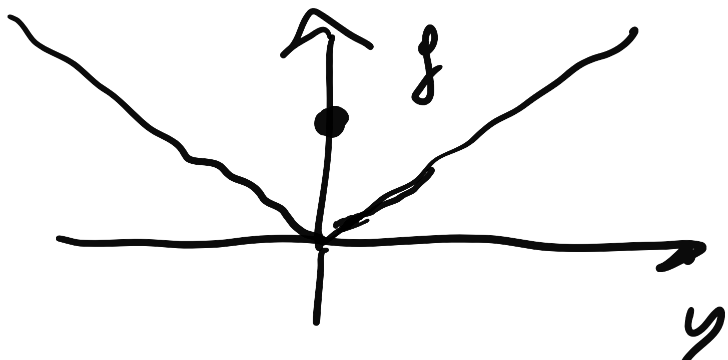
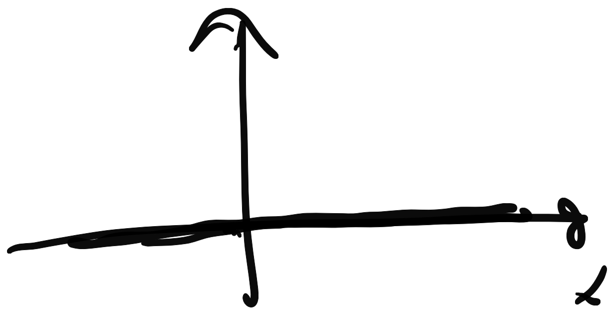
iii) is essential. Indeed

consider $g(y) = \begin{cases} |y| & y \neq 0 \\ 1 & y = 0 \end{cases}$

$h(x) \equiv 0$





ii) $\boxed{\lim_{y\to 0} g(y) = 0}$

i) $\lim_{x\to 0} h(x) = 0$

$\lim_{x\to 0} g(h(x)) \equiv$

$\equiv 1$

Let's check iii) $h(x) \neq y_0$ not verified because $h(x) = 0 = 0 = y_0$

When g is continuous at y_0 we do not need hyp. iii) in the Theorem

Theorem 1 h and g and f as before

i) $\lim_{x \to \xi} h(x) = y_0$

ii) $\lim_{y \to y_0} g(y) = l = g(y_0)$

iii) g is continuous at y_0

Then $\lim f(x) = \lim g(h(x)) = l$
$= g(y_0)$

Let us prove Theorem 0

$\lim_{x \to \xi} f(x) = \lim_{x \to \xi} g(h(x)) =$

$\overset{?}{=} g \circ h(\xi)$

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i) $h(x) \longrightarrow h(\xi) = y_0$ for $x \to \xi$ (because h cont.)

ii) $g(y) \longrightarrow g(y_0)$ for $y \to y_0$

iii) g is continuous at y_0

So we are in the hypotheses of the Theorem 1 (new Theorem on change of variable)

$$\lim_{x\to\xi} f(x) = \lim_{x\to\xi} g(h(x)) = g(y_0) =$$
$$= g(h(\xi)) = f(\xi)$$

Exercise: Consider

$$f(x) = \sin\left(\frac{1}{x^2-3x+2}\right)$$

1) Find the maximal domain.

2) is the function continuous on its domain?

Solution

1) $x \in \mathbb{R}$ $\quad x^2-3x+2 \neq 0$

$$\Updownarrow$$

$$(x-1)(x-2) \neq 0$$

1

$x \neq 1 \quad x \neq 2$

$$D = \boxed{\mathbb{R} \setminus \{1, 2\}}$$

2) **yes** because f is the composition of

$$f = g \circ h$$

$g(y) = \sin y$ cont. on $\mathbb{R}$

$h(x) = \dfrac{1}{x^2 - 3x + 2}$ cont $\forall x \neq 1, 2$

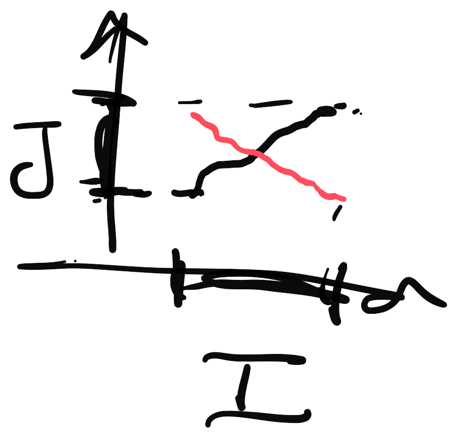
$1, 2 \in Acc(D)$

Does $\lim_{x \to 1} f(x)$ exist?

Does $\lim_{x \to 2} f(x)$ exist?

Theorem. I, J intervals

$$f : I \longrightarrow J$$



1) f is monotonic (that is, increasing or decreasing)

2) f is surjective i.e. $f(I) = J$

Then f is continuous at each $\xi \in I$.

So <u>Corollary</u>:

$f : I \to J$

is i) strictly monotonic

ii) surjective

Then both $f : I \to J$

$f^{-1} : J \to I$

are continuous at each point of their domain.

Hence

a^x $\quad a > 0$ is continuous

$\log_a x$ $\quad a > 0, a \neq 1$ is continuous

x^α is continuous.

$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}$

is strictly monotonic from

$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ onto $J = [-1, 1]$

$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

is continuous

(because it is the inverse of a strictly monoto-)