$$\lim_{x\to 0} \frac{\sqrt[5]{1+3x^4} - \sqrt{1-2x}}{\sqrt[3]{1+x} - \sqrt{1-x}} =$$

$$= \lim_{x\to 0} \frac{(1+3x^4)^{\frac{1}{5}} - (1-2x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{2}}} =$$

is a $\frac{0}{0}$ form

Exploit

$$\frac{(1+y)^{\alpha} - 1}{y} \xrightarrow{y\to 0} \alpha$$

$$= \lim_{x\to 0} \frac{(1+3x^4)^{\frac{1}{5}} - 1 - \left((1-2x)^{\frac{1}{2}} - 1\right)}{(1+x)^{\frac{1}{3}} - 1 - \left((1-x)^{\frac{1}{2}} - 1\right)} =$$

$=$

$$\frac{\dfrac{(1+3x^4)^{\frac{1}{5}}-1}{3x^4}\,3x^4-\dfrac{(1-2x)^{\frac{1}{2}}-1}{2x}\,2x}{\dfrac{(1+x)^{\frac{1}{3}}-1}{x}\,x+\dfrac{(1-x)^{\frac{1}{2}}-1}{-x}\,(-x)}$$

$x \to 0$ $\frac{1}{2}$

$$=\lim_{x\to 0}\frac{\dfrac{(1+3x^4)^{\frac{1}{5}}-1}{3x^4}\,3x^3-\dfrac{(1-2x)^{\frac{1}{2}}-1}{2x}\,2}{\dfrac{(1+x)^{\frac{1}{3}}-1}{x}+\dfrac{(1-x)^{\frac{1}{2}}-1}{-x}}$$

Now $\lim\limits_{x\to 0}\dfrac{(1+3x^4)^{\frac{1}{5}}}{3x^4}\underset{y=3x^4}{=}\lim\limits_{y\to 0}\dfrac{(1+y)^{\frac{1}{5}}}{y}=\dfrac{1}{5}$

$$=\frac{\frac{1}{5}\cdot 0-\frac{1}{2}\cdot 2}{\frac{1}{3}+\left(-\frac{1}{2}\right)}=$$

$$=\frac{-1}{-\frac{5}{6}}=\frac{6}{5}$$

Equivalently:

$$\lim_{x\to 0} \frac{(1+3x^4)^{\frac{1}{5}} - (1-2x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{2}}},$$

$\lim_{y\to 0} \frac{(1+y)^a - 1}{y} = a$

$\Updownarrow$

$(1+y)^a = 1 + ay + o(y)$ for $y \to 0$

$(1+h(x))^a = 1 + ah(x) + o(h(x))$

$$\lim_{x\to 0} \frac{\cancel{1} + \frac{1}{5} 3x^4 + \overbrace{o(x^4)}^{=o(x)} - \cancel{1} + \frac{1}{2}2x + o(x)}{\cancel{1} + \frac{1}{3}x + o(x) - (\cancel{1} - \frac{1}{2}x + o(x))}$$

$\frac{1}{5}3x^4 = o(x)$

$$= \lim_{x\to 0} \frac{x + o(x)}{\frac{5}{6}x + o(\frac{5}{6}x)} = \lim \frac{x}{\frac{5}{6}x} = \frac{6}{5}$$

Proposition 1

$o(x) + o(x) = o(x)$

$f \qquad g$

$\frac{f(x)+g(x)}{x} = \frac{f(x)}{x} + \frac{g(x)}{x}$

$\to 0$

Observe we have the form

$$\lim_{x\to x_0} \frac{f(x) + o(f(x))}{g(x) + o(g(x))}$$

Proposition 1: f and g are functions defined in some neighbourhood of $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = l \in \mathbb{R} \cup \{\pm\infty\}$

then

$$\lim_{x\to x_0} \frac{f(x) + o(f(x))}{g(x) + o(g(x))} = \lim_{x\to x_0} \frac{f(x)}{g(x)} = l$$

Proof of Prop. 1:

$$\lim_{x\to x_0} \frac{f(x)+o(f(x))}{g(x)+o(g(x))} = \lim \frac{f(x)\left(1+\frac{o(f(x))}{f(x)}\right)}{g(x)\left(1+\frac{o(g(x))}{g(x)}\right)} =$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \cdot \frac{\left(1+\frac{o(f(x))}{f(x)}\right) \to 1}{\left(1+\frac{o(g(x))}{g(x)}\right) \to 1} =$$

$$= \lim_{x\to x_0} \frac{f(x)}{g(x)} = l.$$

q.e.d.

Exercise

$$\lim_{x\to 0} \frac{(1-\cos 3x)^2}{x^2(1-\cos x)} = \lim_{x\to 0} \frac{\left(\frac{1-\cos 3x}{(3x)^2}\cdot(3x)^2\right)^2}{x^2\cdot\frac{1-\cos x}{x^2}\cdot x^2} =$$

$$= \lim_{x\to 0} \frac{3^4 \cancel{x^4}\left(\frac{1-\cos 3x}{(3x)^2}\right)^2 \to \left(\frac{1}{2}\right)^2}{\cancel{x^4}\cdot\left(\frac{1-\cos x}{x^2}\right) \to \frac{1}{2}} =$$

$$\lim_{x\to 0} 81 \cdot \frac{1}{4} \cdot 2 = \frac{81}{2}.$$

$$\lim_{y\to 0} \frac{1-\cos y}{y^2} = \frac{1}{2}$$

$$\Updownarrow$$

$$\cos y = 1 - \frac{y^2}{2} + o(y^2)$$

$$\Updownarrow$$

$$1-\cos y = \frac{y^2}{2} + o(y^2)$$

$$\lim_{x\to 0}\frac{(1-\cos 3x)^2}{x^2(1-\cos x)} = \lim_{x\to 0}\frac{\left(\frac{(3x)^2}{2}+o(\ )\right)^2}{x^2\left(\frac{x^2}{2}+o(x^2)\right)}$$

$$= \lim_{x\to 0}\frac{\frac{(3x)^4}{4}+o(x^2)\cdot o(x^2)+\cancel{2}\frac{(3x)^2}{\cancel{2}}\cdot o(x^2)}{\frac{x^4}{2}+x^2\cdot o(x^2)} =$$

$$= \lim_{x\to 0}\frac{\frac{(3x)^4}{4}+o(x^4)+o\left((3x)^2x^4\right)}{\frac{x^4}{2}+o(x^4)}$$

$$= \lim_{x\to 0}\frac{\frac{81x^4}{4}+o\left(\frac{81}{4}x^4\right)}{\frac{x^4}{2}+o\left(\frac{x^4}{2}\right)}$$

$$= \lim_{x\to 0}\frac{\frac{81\cancel{x^4}}{4}}{\frac{\cancel{x^4}}{2}} = \frac{81}{4}\cdot 2 = \frac{81}{2}$$

$$\lim_{y\to 0} \frac{\cos y - 1}{y^2} = -\frac{1}{2}\frac{y^2}{y^2}$$

$$\lim_{y\to 0} \frac{1-\cos y - \frac{1}{2}y^2}{y^2} = 0$$

$$\Updownarrow$$

$$1 - \cos y - \frac{1}{2}y^2 = o(y^2)$$

$$\Updownarrow$$

$$1 - \cos y = \frac{1}{2}y^2 + o(y^2)$$

CONTINUITY.

Def $f : D \longrightarrow \mathbb{R}$

$\xi \in A_{cc}(D) \cap D$

We say that f is continuous at ξ if $\boxed{\lim_{x \to \xi} f(x) = f(\xi)}$

Examples:

1) $f(x) = ax + b$ is continuous at ξ $\forall \xi \in \mathbb{R}$

$$\lim_{x \to \xi} f(x) \overset{?}{=} f(\xi)$$

$$\lim_{x \to \xi} \underline{ax + b} = a\xi + b$$

2) $f(x) = a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0$

is it continuous at every $\xi \in \mathbb{R}$?

Let us prove by induction

" $\forall n \in \mathbb{N}$ a polynomial of degree $\leq n$ is continuous at every $\xi \in \mathbb{R}$ "

By induction:

I) For $n = 1$ yes (see 1)

II) Assume it is true for n and prove it for $n+1$

$$f(x) = a_{n+1} x^{n+1} + a_n x^n + \dots + a_1 x + a_0$$

Let $\xi \in \mathbb{R}$ we want to prove that $\lim_{x \to \xi} f(x) \overset{?}{=} f(\xi)$.

$$\Updownarrow$$

$$\lim_{x \to \xi} f(x) - f(\xi) \overset{?}{=} 0.$$

Prove the particular case

$$\boxed{f(x) = x^{n+1}}$$

$$\lim_{x \to \xi} f(x) - f(\xi) = x^{n+1} - \xi^{n+1} =$$

$$\lim_{x \to \xi} \underbrace{(x - \xi)} \left(x^n + x^{n-1}\xi + x^{n-2}\xi^2 + \dots \dots + x\xi^{n-1} + \xi^n \right)$$

$$= \underbrace{\lim_{x\to\xi}(x-\xi)}_{\downarrow\, 0} \cdot \underbrace{\lim_{x\to\xi}\left(x^n + x^{n-1}\xi + \dots + \xi^n\right)}_{\downarrow\, (n+1)\xi^n}$$

$$= 0 \cdot (n+1)\xi^n = 0$$

General case $f(x) = a_{n+1}\underbrace{\boxed{x^{n+1}}}_{\text{continuous (I)}} + \underbrace{a_n x^n + \dots + a_0}_{\text{continuous (II)}}$

f is the sum of continuous functions:

$f(x) = \mathrm{I}(x) + \mathrm{II}(x)$

$\Longrightarrow$ f is continuous at ξ.

Proposition: if f and g are continuous at ξ, $af(x) + bg(x)$ is continuous at ξ

Proof $\lim_{x\to\xi}\left(af(x) + bg(x)\right) = a\lim_{x\to\xi} f(x) + b\lim_{x\to\xi} g(x)$

$= af(\xi) + bg(\xi)$

q.e.d.

$f(x) = \frac{1}{x}$