$\mathbb{R}^3$ $\quad u = e_1 - 3e_3, \quad V = -2e_2 + e_3, \quad W = 2e_1 + e_2 + e_3, \quad t = e_1 + e_3$ $\quad e_i$ vettori base canonica $\mathbb{R}^3$

$$= \begin{pmatrix}1\\0\\-3\end{pmatrix} \qquad = \begin{pmatrix}0\\-2\\1\end{pmatrix} \qquad = \begin{pmatrix}2\\1\\1\end{pmatrix} \qquad = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

ⓐ Mostrare dip. lineare u, v, w, t

$$\begin{cases} a + 2c + d = 0 \\ -2b + c = 0 \\ -3a + b + c + d = 0 \end{cases} \qquad \begin{cases} a + 4b + d = 0 \\ \boxed{c = 2b} \\ -3a + 3b + d = 0 \end{cases}$$

$$4a + b = 0 \qquad \begin{array}{l} b = -4a \\ c = -8a \end{array}$$

$$a - 16a + d = 0 \qquad -15a + d = 0 \qquad d = 15a$$

$$\left.\begin{array}{l} a = 1 \\ b = -4 \\ c = -8 \\ d = 15 \end{array}\right\} \quad u - 4V - 8W + 15t = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

trovare basi per $U \cap W$ e per $U + W$ e per V e W

ⓑ $U := \langle u, v \rangle, \quad W := \langle w, t \rangle$

$$\tilde{a}\begin{pmatrix}1\\0\\-3\end{pmatrix} + \tilde{b}\begin{pmatrix}0\\-2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}\tilde{a}\\-2\tilde{b}\\-3\tilde{a}+\tilde{b}\end{pmatrix} \Rightarrow \tilde{a} = \tilde{b} = 0 \Rightarrow u \text{ e } v \text{ lin. ind.}$$

base per $U = \{u, v\} \Rightarrow \dim_{\mathbb{R}^3}(U) = 2$ $\qquad$ Analog base per $W = \{w, t\}$ $\quad \dim_{\mathbb{R}^3}(W) = 2$

$$U \cap W : \quad u - 4V - 8W + 15t = 0 = 0_{\mathbb{R}^3} \qquad \underbrace{u - 4V}_{U} = \underbrace{8W - 15t}_{W} = \begin{pmatrix}1\\8\\-7\end{pmatrix} \in U \cap W$$

$$U \cap W \neq \langle 0 \rangle_{\mathbb{R}^3} \qquad \dim(U \cap W) \geq 1$$

Può $\dim(U \cap W) = 2$? Se lo fosse avrei $U \cap W = U = W$

$u \in W$? $\quad u \in W \Leftrightarrow \exists\, n, m$ t.c. $nW + mt = u \Leftrightarrow \begin{pmatrix}2n+m\\n\\n+m\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix}$

$$\underset{\substack{\| \\ 1}}{2n+m} = \underset{\substack{\| \\ 0}}{n} + \underset{\substack{\| \\ -3}}{(n+m)} \qquad 1 = 0 - 3 = -3 \quad ↯$$

$$u \notin W \Rightarrow W \neq U$$
$$\Rightarrow \dim(U \cap W) = 1$$

$\mathbb{R}^3$ $\quad u = e_1 - 3e_3 \quad V = -2e_2 + e_3$

Base per $U \cap W = \left\{ \begin{pmatrix} 1 \\ 8 \\ -7 \end{pmatrix} \right\}$ $\qquad U+W = ?$

$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 2+2-1 = 3$ $\qquad \dim(U+W) = 3 \Rightarrow U+W = \mathbb{R}^3$

Basi per $U+W$: $\{u, v, w\}, \{u, v, t\}, \{e_1, e_2, e_3\}$

~ INTERMEZZO ~ Può $U \cap W = \langle 0 \rangle_{\mathbb{R}^3}$?

$\dim(U+W) = \underset{\underset{2}{\parallel}}{\dim U} + \underset{\underset{2}{\parallel}}{\dim W} - \dim(U \cap W)$ $\qquad$ se $\dim(U \cap W) = 0$ $(\Leftrightarrow U \cap W = \langle 0 \rangle_{\mathbb{R}^3})$

$\Downarrow \dim(U+W) = 2+2-0 = 4$

~ FINE ~

c) $v' = 2e_1 - 3e_2 + e_3 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ $\qquad v' \in U? \in W? \in U \cap W? \in U+W?$

$v' \in U+W = \mathbb{R}^3$ ✓ $\qquad U \cap W = \left\langle \begin{pmatrix} 1 \\ 8 \\ -7 \end{pmatrix} \right\rangle$ $\qquad$ se $v' \in U \cap W$ $\exists a \in \mathbb{R}$ t.c. $a \begin{pmatrix} 1 \\ 8 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$v' \notin U \cap W$

$v' \in U$? Eq. cart. per U: $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + b \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{cases} x_1 = a \\ x_2 = -2b \\ x_3 = -3a + b \end{cases} \right\}$

$b = -\frac{x_2}{2}$ $\qquad x_3 = -3x_1 + \left(-\frac{x_2}{2}\right)$ $\qquad 2x_3 = -6x_1 - x_2$ $\qquad \boxed{6x_1 + x_2 + 2x_3 = 0} : U$

$v' \in U$? $\quad 6 \cdot 2 - 3 + 2 \cdot 1 = 12 - 3 + 2 = 11 \neq 0$ $\quad$ NO $v' \notin U$

Analog $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{cases} x_1 = 2a + b \\ x_2 = a \\ x_3 = a + b \end{cases} \right\}$ $\qquad x_1 = x_2 + x_3$ $\qquad \boxed{x_1 - x_2 - x_3 = 0} : W$

$v' \in W$? $\quad 2 + 3 - 1 = 4 \neq 0$ $\quad$ NO $v' \notin W$

$$\boxed{U \cap W = \begin{cases} 6x_1 + x_2 + 2x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}}$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbb{C}^4$$

$$\begin{cases} a+b=0 \\ b+c=0 \\ -2a-b+c=0 \\ -c=0 \end{cases} \Rightarrow a=b=c=0$$

$\underline{U = \langle u_1, u_2, u_3 \rangle}$

$$\begin{cases} x_1 = a+b \\ x_2 = b+c \\ x_3 = -2a-b+c \\ x_4 = -c \end{cases} \Rightarrow \begin{cases} a = x_1 - x_2 - x_4 \\ b = x_2 + x_4 \\ c = -x_4 \end{cases}$$

$$x_3 = -2x_1 + 2x_2 + 2x_4 - x_2 - x_4 - x_4 =$$
$$= -2x_1 + x_2 \Rightarrow \boxed{U : \ 2x_1 - x_2 + x_3 = 0}$$

↳ 1 equazione $= 4 - 3$, $4 = \dim(\mathbb{C}^4)$, $3 = \dim(U)$

$\boxed{\text{Completare ad una base di } \mathbb{C}^4}$

$\mathbb{C}^4 = \langle u_1, u_2, u_3, e_1 \rangle$

$\boxed{\text{Determinare } W_0 \text{ t.c. } \mathbb{C}^4 = U \oplus W_0}$

$\dim U = 3$

$\dim \mathbb{C}^4 = 4$

$\dim(U \cap W_0) = 0$

$\dim(W_0) = 4 - 3 = 1$

$W_0 = \langle e_1 \rangle$

$\boxed{\text{Determinare \underline{tutti} } W \text{ t.c. } \mathbb{C}^4 = U \oplus W}$

$W = \langle x_1 u_1 + x_2 u_2 + x_3 u_3 + x_0 e_1 \rangle$ $\quad \boxed{x_0 = 1}$

$W \not\subset U \Rightarrow x_0 \neq 0$

$W = \langle \underbrace{x_1 u_1 + x_2 u_2 + x_3 u_3}_{\in U} + e_1 \rangle$