$$\sin(x) = x + o(x) \iff \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2) \iff \lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

$$\log(1+x) = x + o(x) \iff \lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$e^x - 1 = x + o(x) \iff \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim \frac{\log(1+x)}{x} = 1$$

$$l \quad (1+x) - x = o(x)$$

$$\lim_{x \to 0} \frac{\log(1+x) - x}{x} = \lim \underbrace{\frac{\log(1+x)}{x}}_{1} - 1 = 0$$

$$\Downarrow$$

$$\log(1+x) - x = o(x)$$

$$\alpha > 0$$

$$(1+x)^{\alpha} = 1 + \alpha x + o(x)$$

$$\Updownarrow$$

$$\lim \frac{(1+x)^{\alpha} - 1}{x} = \lim_{x \to 0} \frac{e^{\alpha \log(1+x)} - 1}{x}$$

$$\underset{y = \alpha \log(1+x)}{=} \lim_{y \to 0} \frac{e^{y} - 1}{e^{y/\alpha} - 1} =$$

$$\frac{y}{\alpha} = \log(1+x)$$

$$1 + x = e^{y/\alpha}$$

$$\lim_{y \to 0} \underbrace{\frac{e^{y} - 1}{y}} \frac{y}{e^{y/\alpha} - 1} =$$

$$= 1 \cdot \lim_{y \to 0} \frac{y/\alpha}{e^{y/\alpha} - 1} \alpha \underset{z = y/\alpha}{=}$$

$$= 1 \cdot \lim_{z \to 0} \frac{z}{e^{z} - 1} \alpha = 1 \cdot 1 \cdot \alpha = \alpha$$

Equivalently:

$$\lim_{x \to 0} \frac{e^{\alpha \log(x+1)} - 1}{x}$$

$$\left(e^{y} - 1 = y + o(y)\right)$$

$$\lim_{x \to 0} \frac{\alpha \log(x+1) + o(\alpha \log(x+1))}{x}$$

$$= \alpha \lim_{x \to 0} \frac{\log(x+1)}{x} + \left(\frac{o(\alpha \log(x+1))}{x}\right)$$

$$= \alpha + \lim_{x \to 0} \frac{o\left(\alpha\left(x + o(x)\right)\right)}{x}$$

$$= \alpha + \lim \frac{o(x)}{x} =$$

$$= \alpha .$$

Operations with o

$o(h)$ $\quad o(k)$ $\quad$ for $\underline{\underline{x \to x_0}}$

f '' $\qquad g$ ''

$\Updownarrow \qquad \Updownarrow$

$$\lim_{x \to x_0} \frac{f(x)}{h(x)} = 0 \qquad \lim_{x \to x_0} \frac{g(x)}{k(x)} = 0$$

$$o(h) \cdot o(k) \overset{?}{=} o(h \cdot k)$$

$$\lim_{x \to x_0} \frac{f \cdot g}{h \cdot k} = \underset{\overset{\|}{0}}{\lim_{x \to x_0} \frac{f}{h}} \cdot \underset{\overset{\|}{0}}{\lim_{x \to x_0} \frac{g}{k}} =$$

$= 0$

---

b = bounded function in a neighbor. of x_0

$$b \cdot \underbrace{o(h)}_{f} \overset{?}{=} o(b \cdot h)$$

$$\lim_{x \to x_0} \frac{\cancel{b} \cdot f}{\cancel{b}\, h} = 0$$

Exercise

$$\lim_{x\to 0}\frac{\log\left(1+\sin^2 x\right)-\sin^2 x}{\sin^2 x}=$$

$$\frac{\log\left(1+\left(x+o(x)\right)^2\right)-\left(x+o(x)\right)^2}{\left(x+o(x)\right)^2}=$$

$$=\frac{\log\left(\underline{1}+\overbrace{x^2+2x\,o(x)+\left(o(x)\right)^2}^{\to 0}\right)-x^2-2x\,o(x)-\left(o(x)\right)^2}{x^2+2x\,o(x)+\left(o(x)\right)^2}$$

$$=\frac{\cancel{x^2}+\overset{o(x^2)}{\overset{\|}{2x\left(o(x)\right)}}+\overset{o(x^2)}{\overset{\|}{\left(o(x)\right)^2}}-\cancel{x^2}-\overset{o(x^2)}{\overset{\|}{2x\left(o(x)\right)}}-\underset{o(x^2)}{\underset{\|}{\left(o(x)\right)^2}}+o\left(x^2+2x\left(o(x)\right)+\left(o(x)\right)^2\right)}{x^2+2x\,o(x)+\left(o(x)\right)^2}=$$

$$\left(\underbrace{\log(1+y)} = y + o(y)\right)$$

is that true that $o(h) - o(h) = 0$?

$f = x^2 + x^3$

$g = x^4 + x^3$

$f - g \neq 0$

$\shortparallel$

$x^2 + \cancel{x^3} - x^4 - \cancel{x^3}$

$f = o(x)$

$\lim \frac{f(x)}{x} = \lim \frac{x^2+x^3}{x} = 0$

$g = o(x)$. ——

$f - g = o(x)$

$o(h) \pm o(h) = o(h)$

$$\frac{o(x^2)}{x^2 + o(x^2) + o(x^2)} =$$

$$= \lim_{x\to 0} \frac{o(x^2)}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{\overset{\nearrow 0}{\frac{o(x^2)}{x^2}}}{1 + \underset{\downarrow\atop 0}{\frac{o(x^2)}{x^2}}}$$

$$= \frac{0}{1} = 0$$

New functions !!

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

Why these names?

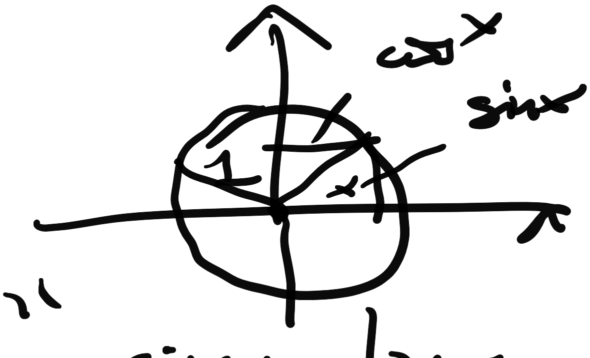
$\cos x$

$\sin x$

$$X^2 + Y^2 = 1$$

"

$$(\cos x)^2 + (\sin x)^2 = 1$$



" circular "
functions

Observe that

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

$$\parallel$$

$$\frac{(e^x+e^{-x})^2}{4} - \frac{(e^x-e^{-x})^2}{4} =$$

$$\frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2}{4} =$$

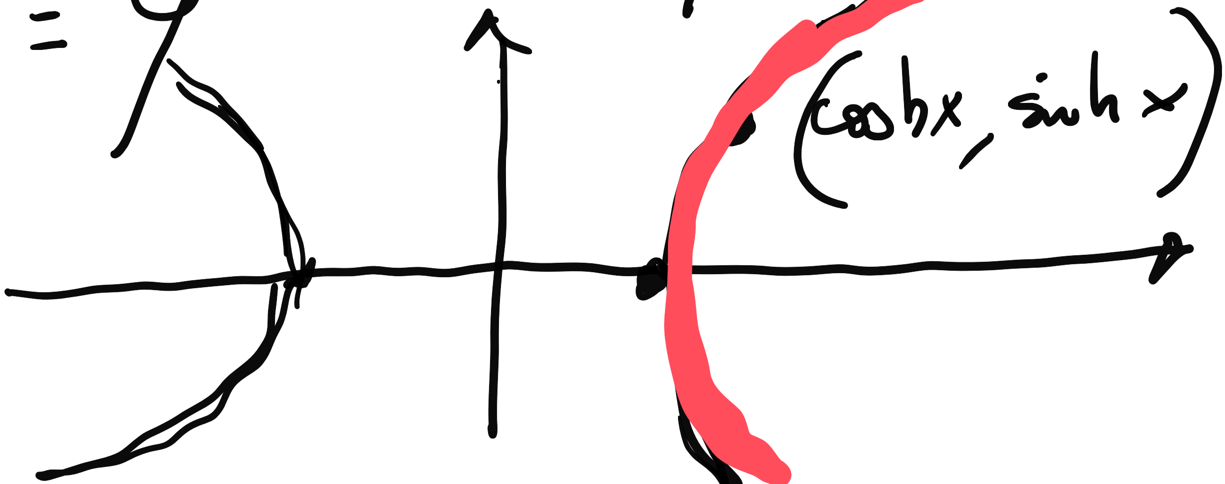
$$\frac{4}{4} = 1$$

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

$\cosh x = X$

$\sinh x = Y$

$$\boxed{X^2 - Y^2 = 1}$$



$$\boxed{\lim_{x\to 0}\frac{\sinh x}{x}=1}\iff\lim_{x\to 0}\frac{\sinh x}{x}-1=0$$

$$\lim_{x\to 0}\frac{\frac{e^{x}-e^{-x}}{2}-1}{x}=$$

$$=\lim_{x\to 0}\frac{\frac{e^{x}-e^{-x}-2x}{2}}{x}\underset{o(x)}{=}$$

$$=\lim_{x\to 0}\frac{\frac{1+x+o(x)-\left(1-x+o(-x)\right)-2x}{2}}{x}$$

$$\approx\lim_{x\to 0}\frac{\frac{\cancel{2x}+o(x)-\cancel{2x}}{2}}{x}=$$

$$=\lim_{x\to 0}\frac{\frac{o(x)}{2}}{x}=$$

$$=\lim_{x\to 0}\frac{o(x)}{x}=\boxed{0}$$

$$\lim_{x\to 0} \frac{\cosh(x)-1}{x^2} =$$

$$\lim_{x\to 0} \frac{(\cosh(x)-1)(\cosh(x)+1)}{x^2(\cosh(x)+1)} =$$

$$= \lim_{x\to 0} \frac{\cosh^2(x)-1}{x^2(\cosh(x)+1)} =$$

$$= \lim_{x\to 0} \underset{\underset{\displaystyle \underline{1}}{\swarrow}}{\frac{\sinh^2(x)}{x^2}} \cdot \underset{\underset{\underset{\displaystyle \frac{1}{2}}{\searrow}}{\displaystyle \frac{1}{\cosh(0)+1}}}{\underset{\swarrow}{\frac{1}{\cosh(x)+1}}}$$

$$= 1 \cdot \frac{1}{2} = \frac{1}{2}$$

<u>Hence</u>:

$$\lim_{x\to 0} \frac{\cosh x - 1}{x^2} = \frac{1}{2}$$

$$\Updownarrow$$

$$\lim_{x\to 0}\frac{\cosh x - 1 - \frac{1}{2}x^2}{x^2} = 0$$

$$\cosh x - 1 - \frac{1}{2}x^2 = o(x^2)$$

$$\cosh x = 1 + \frac{1}{2}x^2 + o(x^2)$$

$$\underline{\cos x} \qquad = \boxed{1 - \frac{1}{2}x^2} + o(x^2)$$

Observe: cosh is even

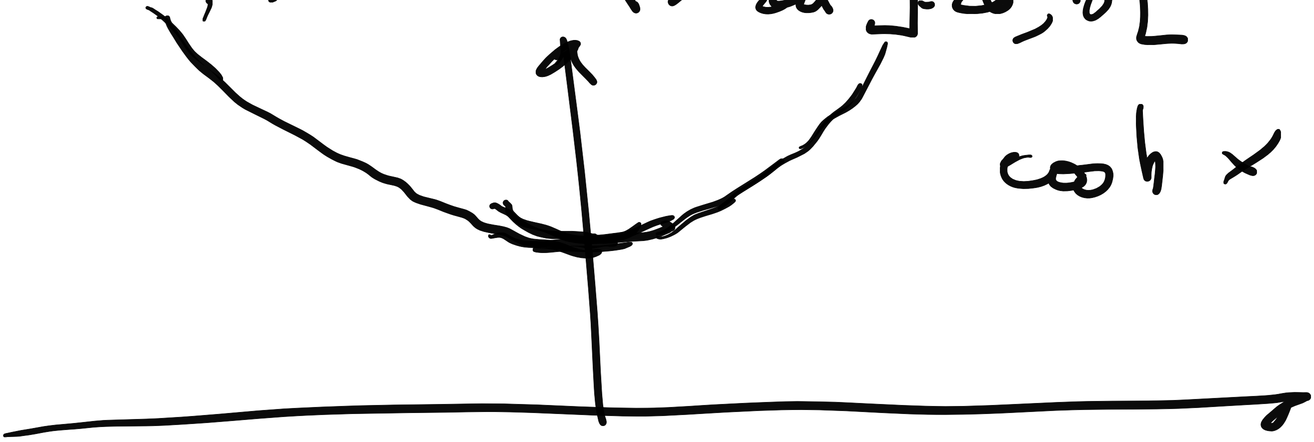
i.e. $$\cosh(x) = \cosh(-x)$$

$$\frac{e^x+e^{-x}}{2} \qquad \frac{e^{-x}+e^{x}}{2}$$

Prove: sinh is increasing

and $\sinh(x) = -\sinh(x) \quad \forall x$

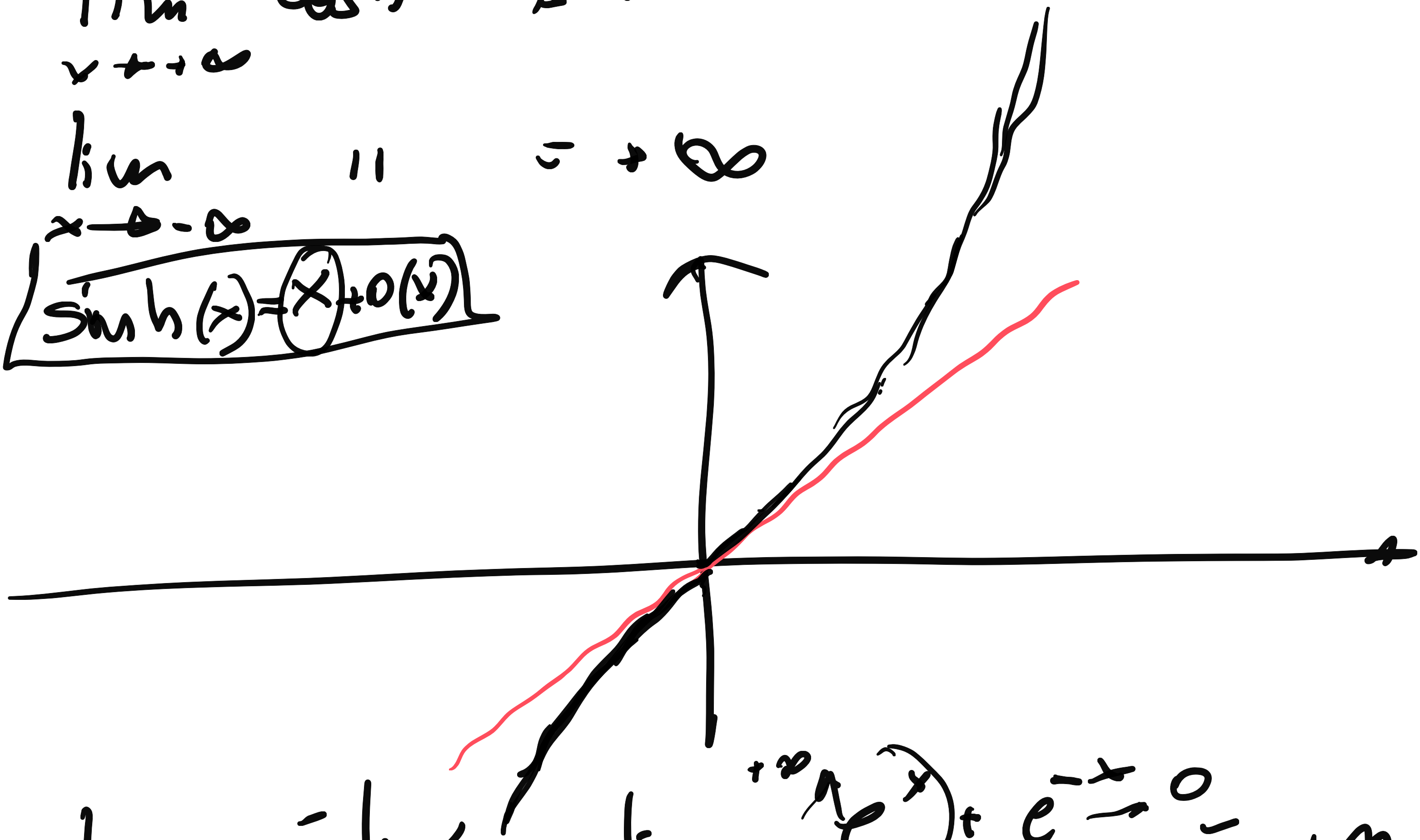
Deduce that: cosh x is increasing on $[0,+\infty[$ $\Rightarrow$ it is decreasing on $]-\infty,0[$



$$\lim_{x\to+\infty} \cosh x = +\infty$$

$$\lim_{x\to-\infty} \quad '' \quad = +\infty$$

$$\boxed{\sinh(x) = x + o(x)}$$



$$\lim_{x\to+\infty} \sinh x = \lim_{x\to+\infty} \frac{\overset{+\infty}{e^{x}} + e^{-x}\to 0}{2} = +\infty$$

Exercise

$$\lim_{x\to 0} \frac{\log(2-\cos x)}{\cosh(x)-1} =$$

$$= \lim_{x\to 0} \frac{\log(1+(1-\cos x))}{\cosh(x)-1} =$$

$$= \lim_{x\to 0} \frac{(1-\cos x)+o(1-\cos x)}{\frac{1}{2}x^2+o(x^2)} =$$

$$= \lim_{x\to 0} \frac{\frac{1}{2}x^2+\overbrace{o(x^2)+o(x^2)}^{o(x^2)}}{\frac{1}{2}x^2+o(x^2)} =$$

$$= \lim_{x\to 0} \frac{\frac{1}{2}+\frac{o(x^2)}{x^2} \rightarrow 0}{\frac{1}{2}+\frac{o(x^2)}{x^2} \rightarrow 0}$$

$$= \underline{1}$$