Lesson 17 - 07/11/2022

- Kinetic energy in terms of $\vec{q} = (q_1 — q_m)$
- 4 examples
- Def. of $Q_h := \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} \quad \forall\ h = 1 — m$
- Conservative case :
- Towards Lagrange eqs...

Constrained system of $P_1 — P_N$ points, described by $\vec{q} = (q_1 — q_m)$ Lagrangian coordinates.

$$K_i = \frac{1}{2} m_i |\vec{v}_i|^2 \quad \text{where} \quad \vec{v}_i = \dot{\vec{OP}}_i$$

$\downarrow$

without constraint

If we have a constrained system of N points, $\vec{OP}_i = \vec{OP}_i(\vec{q}, t)$

$$\Rightarrow \vec{v}_i = \dot{\vec{OP}}_i(\vec{q}, t) = \sum_{h=1}^{m} \frac{\partial \vec{OP}_i}{\partial q_h}(\vec{q}, t)\, \dot{q}_h + \frac{\partial \vec{OP}_i}{\partial t}(\vec{q}, t)$$

Therefore, K = total kinetic energy of the constrained system of $P_1 — P_N$ points is given by:

$$K = \frac{1}{2} \sum_{i=1}^{N} m_i |\vec{v}_i|^2 =$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \left[\sum_{h,k=1}^{m} \frac{\partial \vec{OP}_i}{\partial q_h} \frac{\partial \vec{OP}_i}{\partial q_k} \dot{q}_h \dot{q}_k + \right.$$

$$\left. + 2 \sum_{h=1}^{m} \frac{\partial \vec{OP}_i}{\partial q_h} \cdot \frac{\partial \vec{OP}_i}{\partial t} \dot{q}_h + \frac{\partial \vec{OP}_i}{\partial t} \cdot \frac{\partial \vec{OP}_i}{\partial t} \right] =$$

$$= \frac{1}{2} \sum_{h,k=1}^{m} \underbrace{\left(\sum_{i=1}^{N} m_i \frac{\partial \vec{OP}_i}{\partial q_h} \cdot \frac{\partial \vec{OP}_i}{\partial q_k} \right)}_{a_{hk} = a_{hk}(\vec{q}, t)} \dot{q}_h \dot{q}_k +$$

$$+ \sum_{h=1}^{m} \underbrace{\left(\sum_{i=1}^{N} m_i \frac{\partial \vec{OP}_i}{\partial q_h} \frac{\partial \vec{OP}_i}{\partial t} \right)}_{b_h = b_h(\vec{q}, t)} \dot{q}_h + \underbrace{\frac{1}{2} \sum_{i=1}^{N} m_i \frac{\partial \vec{OP}_i}{\partial t} \cdot \frac{\partial \vec{OP}_i}{\partial t}}_{c = c(\vec{q}, t)} =$$

$$= \frac{1}{2} \sum_{h,k=1}^{m} a_{hk} \dot{q}_h \dot{q}_k + \sum_{h=1}^{m} b_h \dot{q}_h + \frac{1}{2} C =$$

$$= K_2 + K_1 + K_0$$

Remark The matrix (a_{hk}) is symmetric & positive definite.

- (a_{hk}) is (clearly) symmetric, since the scalar product is symmetric. $[a_{hk} = a_{kh} \ \forall h,k = 1 \ldots n]$
- To prove that (a_{hk}) is positive def, we consider:

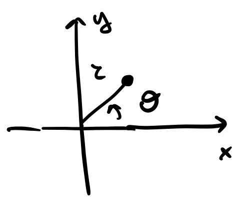
$$\vec{v}_i^{\,*} = \sum_{h=1}^{m} \frac{\partial \vec{P}_i}{\partial q_h} \dot{q}_h$$

has rk max $= m$.

Then: $\frac{1}{2} \sum_{h,k=1}^{m} a_{hk} \dot{q}_h \dot{q}_k = \frac{1}{2} \sum_{i=1}^{N} m_i |\vec{v}_i^{\,*}|^2$

By the cond. of max rk, we obtain that if $\dot{\vec{q}} \neq 0$ then $\vec{v}_i^{\,*} \neq 0 \Rightarrow > 0$.

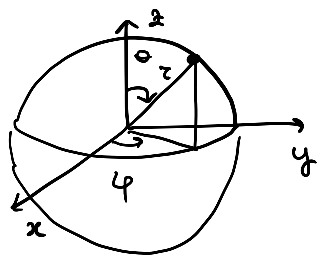
Ex 1 Kinetic energy of a point in polar and spherical coordinates.



$\theta \in [0, 2\pi[$
$r \in (0, +\infty)$

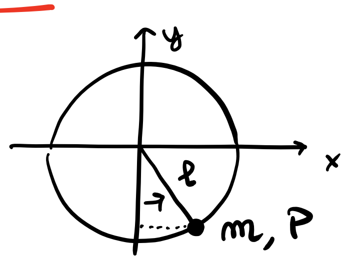
$$\begin{cases} x = r\cos\theta \rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ y = r\sin\theta \rightarrow \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$$

$$K = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow K = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

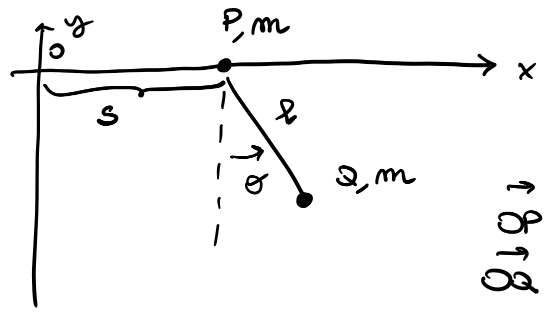
Ex 2 Pendulum



$$\begin{cases} x = \ell\sin\theta \\ y = -\ell\cos\theta \end{cases} \rightarrow \begin{cases} \dot{x} = \ell\dot{\theta}\cos\theta \\ \dot{y} = \ell\dot{\theta}\sin\theta \end{cases}$$

$$K = \frac{1}{2} m \ell^2 \dot{\theta}^2$$

Ex 3



s distance of P from O.

θ angle as in the picture.

$(N=2)$

$\vec{OP} = (s, 0) \Rightarrow |\vec{v}_P|^2 = \dot{s}^2$

$\vec{OQ} = (s + \ell\sin\theta, -\ell\cos\theta)$

$\vec{v}_Q = (\dot{s} + \ell\dot{\theta}\cos\theta, \ell\dot{\theta}\sin\theta)$

$$|\vec{v}_Q|^2 = \dot{s}^2 + \underbrace{\ell^2\dot{\theta}^2\cos^2\theta} + 2\ell\cos\theta\,\dot{s}\dot{\theta} + \underbrace{\ell^2\dot{\theta}^2\sin^2\theta}$$

$$= \dot{s}^2 + \ell^2\dot{\theta}^2 + 2\ell\cos\theta\,\dot{s}\dot{\theta}$$

$$K = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}m\left[\dot{s}^2 + \ell^2\dot{\theta}^2 + 2\ell\cos\theta\,\dot{s}\dot{\theta}\right]$$

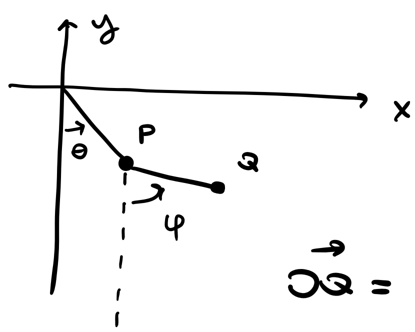
$$= \frac{1}{2}m\left[2\dot{s}^2 + \ell^2\dot{\theta}^2 + 2\ell\cos\theta\,\dot{s}\dot{\theta}\right]$$

$$a = m\begin{pmatrix} 2 & \ell\cos\theta \\ \ell\cos\theta & \ell^2 \end{pmatrix}$$

$(\dot{s}, \dot{\theta})$

$$K = \frac{1}{2}(\dot{s}, \dot{\theta})\, a \begin{pmatrix} \dot{s} \\ \dot{\theta} \end{pmatrix}$$

Ex 4



ℓ, m of points P, Q.

$$K_P = \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$\vec{OQ} = (\ell\sin\theta + \ell\sin\varphi, -\ell\cos\theta - \ell\cos\varphi)$$

$$\vec{v}_Q = \ell(\dot{\theta}\cos\theta + \dot{\varphi}\cos\varphi, \dot{\theta}\sin\theta + \dot{\varphi}\sin\varphi)$$

$$\Rightarrow |\vec{v}_Q|^2 = \ldots = \ell^2\left[\dot{\theta}^2 + \dot{\varphi}^2 + \underbrace{2\cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + 2\sin\theta\sin\varphi\,\dot{\theta}\dot{\varphi}}_{= 2\cos(\theta-\varphi)\dot{\theta}\dot{\varphi}}\right]$$

$$\Rightarrow K_Q = \frac{1}{2} m e^2 \left[\dot\vartheta^2 + \dot\varphi^2 + 2\cos(\vartheta-\varphi)\dot\vartheta\dot\varphi \right]$$

$$K = K_P + K_Q = \frac{1}{2} m e^2 \left[2\dot\vartheta^2 + \dot\varphi^2 + 2\cos(\vartheta-\varphi)\dot\vartheta\dot\varphi \right]$$

$$\mathcal{A} = m e^2 \begin{pmatrix} 2 & \cos(\vartheta-\varphi) \\ \cos(\vartheta-\varphi) & 1 \end{pmatrix}$$

$$K = \frac{1}{2} (\dot\vartheta, \dot\varphi)\, \mathcal{A} \begin{pmatrix} \dot\vartheta \\ \dot\varphi \end{pmatrix}$$

—— x —— x ——

Towards Lagrange eqs .... $P_1 — P_N$ points.

$$\sum_{i=1}^{N} \underbrace{m_i \vec{a}_i} = \sum_{i=1}^{N} \underbrace{(\vec{F}_i + \vec{\phi}_i)}$$

$$\Downarrow$$

$$\sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} = \sum_{i=1}^{N} \underbrace{(\vec{F}_i + \vec{\phi}_i) \cdot \frac{\partial \vec{OP}_i}{\partial q_h}}_{\equiv 0 \quad \forall h = 1 - m}$$

Since the constrain is ideal

$$\Downarrow$$

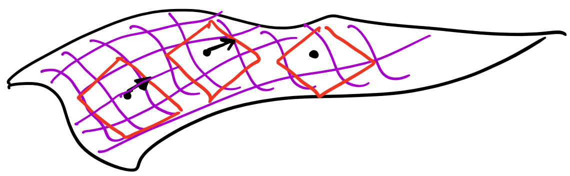
$$\sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} = \underbrace{\sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h}}_{Q_h} \qquad \forall h = 1 - m$$

Def $Q_h = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} \qquad \forall h = 1 - m$

↓

Lagrangian component of external forces

Previous equality becomes : $\boxed{\sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h}} = Q_h \quad \forall h = 1 - m.$



Conservative case

$$\vec{F}_i = -\nabla_i V(\vec{OP}_1 \ldots \vec{OP}_N)$$

In terms of $\vec{q}$: $\hat{V}(\vec{q}) = V(\vec{OP}_1(\vec{q}), \ldots, \vec{OP}_N(\vec{q}))$.

We obtain :

$$Q_h = \sum_{i=1}^{N} -\nabla_i V \cdot \frac{\partial \vec{OP}_i}{\partial q_h} = -\frac{\partial \hat{V}}{\partial q_h} \qquad \forall\, h = 1 \ldots m$$

By the composition rule for derivation

THEOREM (Lagrange eqs).

A constrained system of $P_1 \ldots P_N$ points, with Lagrangian coordinates $q_1 \ldots q_m$.

$\vec{F}_i$, $i = 1 \ldots N$, ideal constrain

Then :

$$\boxed{\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h}} = Q_h \qquad \forall\, h = 1 \ldots m$$

PROOF Next monday !!

Remark In the conservative case : $Q_h = -\dfrac{\partial \hat{V}}{\partial q_h}$

Lagrange eqs becomes :

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} = \underbrace{-\frac{\partial \hat{V}}{\partial q_h}} \iff$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_h} - \frac{\partial}{\partial q_h}\left[K - \hat{V}\right] = 0 \iff$$

$$L = K - \hat{V}$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_h} - \frac{\partial L}{\partial q_h} = 0 \qquad \forall\, h = 1 - n}$$

In the conservative case

Since $\hat V$ doesn't depend on $\dot q$ $\Rightarrow$ $\frac{\partial}{\partial \dot q_h}\hat V \equiv 0$

In the "mixed" case $\rightarrow$ conservative forces $Q_h = -\frac{\partial \hat V}{\partial q_h}$

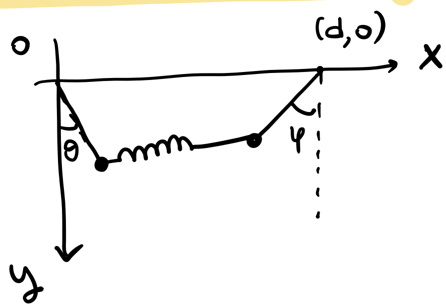
$\searrow$ non-conserv. forces Q_h'

$$L = K - \hat V$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_h} - \frac{\partial L}{\partial q_h} = Q_h'}$$

In the general case

EX 5



On the plane Oxy

ℓ = lenghts

P, m

Q, m

$K>0$ spring constant

1) Det. Lagrangian of the system

2) Det. eqs. of motion.

$$\vec{OP} = (\ell \sin\theta,\ \ell\cos\theta)$$

$$\vec{OQ} = (d - \ell\sin\varphi,\ \ell\cos\varphi)$$

$$\vec v_P = (\ell\dot\theta\cos\theta,\ -\ell\dot\theta\sin\theta) \rightarrow |\vec v_P|^2 = \ell^2\dot\theta^2$$

$$\vec v_Q = (-\ell\dot\varphi\cos\varphi,\ -\ell\dot\varphi\sin\varphi) \rightarrow |\vec v_Q|^2 = \ell^2\dot\varphi^2$$

$$K = \frac{1}{2} m \ell^2 (\dot\theta^2 + \dot\varphi^2)$$

$$V = \frac{1}{2} K |\vec{PQ}|^2$$

$$|\vec{PQ}|^2 = d^2 + 2\ell^2 - 2\ell^2\cos(\theta - \varphi) - 2d\ell(\sin\theta + \sin\varphi)$$

$$V = \frac{1}{2} k |\vec{PQ}|^2$$

$$L = K - V \quad \text{(up to constants)}$$

(• if the plane is vertical, we also need to add the contribution of gravitational conservative force!)

—x—x—

$$\boxed{m \ell \ddot{\theta} = -mg \sin\theta}$$

$$s = \ell\theta \qquad \ddot{s} = \ell\ddot{\theta}$$

$$-mg \sin\left(\frac{s}{\ell}\right) = -\frac{\partial}{\partial s}\left(V(s)\right)$$

$$V(\theta) = -mg\ell\cos\theta$$

$$= -\frac{\ell}{\ell} mg \sin\left(\frac{s}{\ell}\right) = -\frac{\partial}{\partial s}\left(-\ell mg\cos\left(\frac{s}{\ell}\right)\right)$$

$$\Downarrow$$

$$V(s) = -\ell mg \cos\left(\underbrace{\frac{s}{\ell}}_{=\theta}\right)$$