1) QUESTIONS ABOUT PREVIOUS EXERCISES

2) LIMITS WITH "LITTLE-O" METHOD

$\boxed{x \to 0}$

$$\lim_{x\to+\infty} e^{\frac{1}{x}} = 1 + \frac{1}{x} + o\left(\frac{1}{x}\right)$$

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + o(x^n) \\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n) \\
(1+x)^a &= 1 + ax + \frac{a(a-1)}{2}x^2 + \cdots + \binom{a}{n}x^n + o(x^n) \quad \text{dove } \binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \\
\tan x &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots + o(x^6) \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \\
\tanh x &= x - \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^6) \\
\arcsin x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots + \frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} + o(x^{2n+2}) \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \\
\operatorname{arsinh} x &= x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots + \frac{(-1)^n(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} + o(x^{2n+2}) \\
\operatorname{artanh} x &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})
\end{aligned}
$$

$$e^x = 1 + o(1) \overset{!}{=} 1 + x + o(x) \overset{!}{=} \ ---$$

$$\lim_{x\to 0} \frac{3\operatorname{arctg} x + (1-\cos 2x)\sin^2 x}{27x^4 + 5\sin x}$$

$$(\sin x)^2 = (x + o(x))^2 \overset{!}{=} x^2 + \cancel{2x}\,o(x) + o(x^2) \overset{!}{=} x^2 + o(x^2)$$

$$\cos(f(x)) = 1 - \underbrace{\frac{(f(x))^2}{2}} + \underbrace{o(f^2(x))}$$

$$N: 3(x + o(x)) + \left[\cancel{1} - \left(\cancel{1} - \frac{(2x)^2}{2} + o[(2x)^2]\right)\right](x^2 + o(x^2))$$

$$3(x+o(x)) = 3x + o(3x), \quad o(3x) = o(x)$$

$$\left[\cancel{1} - \left(\cancel{1} - \frac{(2x)^2}{2} + o[(2x)^2]\right)\right] = 2x^2 + o(x^2)$$

$$3x + o(x) + (2x^2 + o(x^2))(x^2 + o(x^2))$$

$$2x^2(x^2 + o(x^2)) + o(x^2)(x^2 + o(x^2))$$

$4x^4 + o(x^4) + o(x^4) + o(x^4)$

$\underbrace{4x^4}+\underbrace{o(x^4)} = \underline{o(x^3)} = o(x^2) = o(x)$

$3x + o(x) + 4x^4 + o(x^4)$

$o(x^3)$

$$\lim_{x\to\infty} \frac{1+x}{x} = \lim_{x\to\infty} \frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\,1} = \lim_{x\to\infty} 1+\underbrace{\frac{1}{x}}_{\to 0} = 1$$

D: $27x^4 + 5\sin x$

$27x^4 + 5x + o(x)$

$5x + o(x)$

$\sin x = x + o(x)$

$x^4 = o(x)$

$o(1) = o(x^0),\ o(x),\ x(x^1),\ \ldots$

$$\lim_{x\to 0} \ldots \lim_{x\to 0} \frac{N}{D} = \lim_{x\to 0} \frac{3x+o(x)}{5x+o(x)} = \lim_{x\to 0} \frac{\cancel{x}\,3+o(1)}{\cancel{x}\,5+o(1)} = \boxed{\frac{3}{5}}$$

$$\lim_{x\to 0} \frac{\sin^2 x - 1 + \cos x + 3\log(1+x)}{x\log(1+x) - 2\sin x + 1 - \cos x}$$

N: $\sin^2 x = x^2 + o(x^2)$ ✓

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$ ✓

$\log(1+x) = x + o(x)$

$\Rightarrow \underbrace{x^2 + o(x^2)} \cancel{-1} \cancel{+1} \underbrace{- \frac{x^2}{2} + o(x^2)} + \underbrace{3x + o(x)}$

$3x + \cancel{\frac{x^2}{2}} + o(x) + \cancel{o(x^2)}$

$3x + o(x)$

D: $x\left(x+o(x)\right) - 2\left(x+o(x)\right) + 1 - \left(1 - \frac{x^2}{2} + o(x^2)\right)$

$1 + o(1)$

$x^2 + o(x^2) - 2x + o(x) + \frac{x^2}{2} + o(x^2)$

$-2x + o(x)$

$\frac{o(x^2)}{x^3} = ?$

e.g. $x^2 = o(x)$

$\lim_{x\to 0} \frac{x^2}{x} = 0$

$x^2 = o(3x)$ ✓

$\lim_{x\to 0} \frac{x^2}{3x} = 0$

e.g. $x^{1+\beta} = o(x)$ $\quad \beta > 0$

$$\lim_{x\to 0}\frac{N}{D}=\lim_{x\to 0}\frac{3x+o(x)}{-2x+o(x)}=\boxed{-\frac{3}{2}}$$

e.g. $x^{1+\beta}=o(x)$ $\quad \beta>0$

$o(N)\,,\ o(N^2)$

4) $\displaystyle\lim_{x\to 0}\frac{\sin x-\operatorname{tg}x}{\operatorname{arctg}x-\operatorname{tg}x}$

N: $x+o(x)-(x+o(x))$

$o(x)$

$\cancel{x}-\frac{x^3}{6}+o(x^3)-\left(\cancel{x}+\frac{x^3}{3}+o(x^3)\right)$

$-\left(\frac{1}{6}+\frac{1}{3}\right)x^3+o(x^3)$

$-\frac{1}{2}x^3+o(x^3)$

D: $x+o(x)-(x+o(x))$

$o(x)$

$\cancel{x}-\frac{x^3}{3}+o(x^3)-\left(\cancel{x}+\frac{x^3}{3}+o(x^3)\right)$

$-\frac{2}{3}x^3+o(x^3)$

$$\to\ \lim\frac{N}{D}=\lim\frac{-\frac{1}{2}x^3+o(x^3)}{-\frac{2}{3}x^3+o(x^3)}=\boxed{3}$$

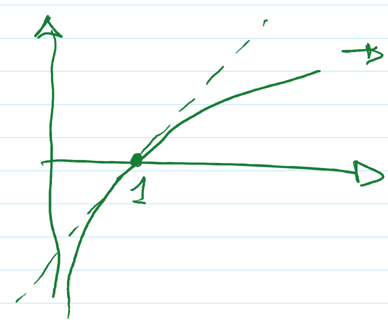
8) $\displaystyle\lim_{x\to 0}\frac{e^{\sin x}+\log\left(\frac{1-x}{e}\right)}{\sin x^2}$

N: $1+\sin x+o(\sin x)+\log(1+(-x))-\log e$

$\sin x = x+o(x)$, $\quad o(\sin x)=o(x+o(x))=o(x)$, $\quad \log(1+(-x))=(-x)+o(x)$, $\quad \log e = 1$

$\cancel{1}+\cancel{x}+o(x)-\cancel{x}+o(x)\cancel{-1}$

$o(x)$



$\cancel{1}+\sin x+\frac{(\sin x)^2}{2}+o(\sin^2 x)+(-x)-\frac{(-x)^2}{2}+o(x^2)\cancel{-1}$

$\sin x = x-\frac{x^3}{6}+o(x^3) = x+o(x^2)$

$o(\sin^2 x)=o(x^2)$

$\frac{(\sin x)^2}{2}\to\frac{(x+o(x))^2}{2}=\frac{x^2}{2}+o(x^2)$

$\cancel{x} + o(x^2) + \cancel{\frac{x^2}{2}} + o(x^2) \cancel{-x} - \cancel{\frac{x^2}{2}} + o(x^2)$

$o(x^2) \longrightarrow -\frac{1}{2}x^3 + o(x^3)$

D: $\sin x^2 = x^2 + o(x^2)$

$\lim \frac{N}{D} = \lim \frac{o(x^2)}{x^2 + o(x^2)} = \boxed{0}$ $\left[= \lim_{x \to 0} -\frac{1}{2}x + o(x) \right]$

10) $\lim_{x \to 0} \frac{x^{\sin x} - 1 - x \log x}{\operatorname{tg} x^2 \log^2 x}$

N: $e^{\underbrace{\sin x \log x}_{\to 0}} - 1 - x \log x$

$\left(1 + \sin x \log x + o(\sin x \log x) - 1 - x \log x \right)$

$\cancel{1} + \underset{\substack{\| \\ x - \frac{x^3}{6} + o(x^3)}}{\sin x \log x} + \underbrace{\frac{(\sin x \log x)^2}{2}}_{\to x + o(x)} + \underbrace{o(\sin^2 x \log^2 x)}_{\to \sim x^2} \cancel{-1} - x \log x$

$x \cancel{\log x} - \cancel{\frac{x^3}{6} \log x} + \cancel{o(x^3 \log x)} + \frac{x^2 \log^2 x}{2} + o(x^2 \log^2 x)$

$+ o(x^2 \log^2 x) - x \cancel{\log x}$

$\log^n x = o(x)$

$\frac{x^2 \log^2 x}{2} + o(x^2 \log^2 x)$

D: $\underset{\substack{\| \\ x^2 + o(x^2)}}{\operatorname{tg} x^2} \log^2 x = x^2 \log^2 x + o(x^2 \log^2 x)$

$\lim \frac{N}{D} = \lim \frac{\frac{1}{2} x^2 \log^2 x + o(x^2 \log^2 x)}{x^2 \log^2 x + o(x^2 \log^2 x)} = \boxed{\frac{1}{2}}$

9) $\lim \frac{e^x \sin x - \frac{x^2}{1+x} - x}{\ldots}$

9) $\lim_{x\to 0} \dfrac{e^x \sin x - \frac{x^2}{1+x} - x}{\operatorname{arctg}^3 x}$

$$N: \left[1+x+\frac{x^2}{2}+o(x^2)\right]\left[x-\frac{x^3}{6}+o(x^3)\right] - x^2\underbrace{(1+x)^{-1}}_{1-x+o(x)} - x$$

$$\cancel{x}-\frac{x^3}{6}+o(x^3)+\cancel{x^2}+o(x^3)+\frac{x^3}{2}+o(x^3)+o(x^3)-\cancel{x^2}+x^3+o(x)-\cancel{x}$$

$$\frac{4}{3}x^3+o(x^3)$$
$$=\left(1+\frac{1}{2}-\frac{1}{6}\right)$$

$$D: \quad x^3+o(x^3)$$

$$\longrightarrow \lim_{x\to 0}\frac{N}{D} = \lim_{x\to 0}\frac{\frac{4}{3}x^3+o(x^3)}{x^3+o(x^3)} = \boxed{\frac{4}{3}}$$

CALCULATE THE LIMITS

1) $\lim_{x\to 0} \dfrac{\sin x^2 + \log(1+2x)}{x\cos x + \sin 3x}$

2) $\lim_{x\to 0^+} \dfrac{4x^2 \sin\sqrt{x} + (1-\cos x)^2}{\sqrt{x}\sinh x^2 + (e^x-1)^3}$

3) $\lim_{x\to a^+} (e^x - e^a)^{\frac{1}{\log(\sin(x-a))}}$

4) $\lim_{x\to 0} \dfrac{\sin x - \operatorname{tg} x}{\operatorname{arctg} x - \operatorname{tg} x}$

5) $\lim_{x\to 0^+} \dfrac{\log(1+2\sqrt{x}\operatorname{arctg} x) - e^{x^2}+1}{\sqrt{1+x^{3/2}}-1}$

6) $\lim_{x\to 0} \dfrac{\operatorname{tg} x - \operatorname{arctg} x}{\log(1+x) - \sin x + x(1-\cos x)}$

7) $\lim_{x\to 0^+} \dfrac{x^{\log x} + \operatorname{ctg}^2 x}{e^{\frac{1}{x}} + (\tanh^2 x)^{-1}}$

8) $\lim_{x\to 0} \dfrac{e^{\sin x} + \log\left(\frac{1-x}{e}\right)}{\sin x^2}$

9) $\lim_{x\to 0} \dfrac{e^x \sin x - \frac{x^2}{1+x} - x}{\operatorname{arctg}^3 x}$

11) $\lim_{x\to 0^+} x\left(x\sin\frac{1}{x} - \log\left(\frac{x^2}{x^2+1}\right) - 1\right)$

12) $\lim_{x\to +\infty} x\left(e^{\frac{2x+2}{x+5}} - e^2\right)$

13) $\lim_{x\to +\infty} (x-\sin x)\left(\frac{1}{x} - \sin\frac{1}{x}\right)$

14) $\lim_{x\to 0} \dfrac{1+\log^2(1+x) - e^{x^2}}{x(1-\cos x)}$

15) $\lim_{x\to 0} \dfrac{\sin x}{\log^2(1+x)} - \dfrac{1}{x}$

16) $\lim_{x\to 0} \dfrac{\sin(\operatorname{arctg}(\sin x)) - \operatorname{arctg}(\sin x)}{\sin(\operatorname{arctg} x) - \sin x}$

17) $\lim_{x\to 0} \dfrac{\sinh(x^x-1)}{1-\cosh(x^x-1-x\log x)}$

18) $\lim_{x\to 0} \dfrac{\log(\sin x) - \log(x\cos x)}{\operatorname{tg} x^2 + e^{-1/x}}$

19) $\lim_{x\to +\infty} x^2\, \dfrac{\log(\sin^2 x) - \log x^2 - \log x}{x - \sin x}$

9) $$\lim_{x \to 0} \frac{e^{\sin x} - \sqrt{1+x} - x}{\operatorname{arctg}^3 x}$$

10) $$\lim_{x \to 0} \frac{x^{\sin x} - 1 - x \log x}{\operatorname{tg} x^2 \log^2 x}$$

19) $$\lim_{x \to +\infty} x^2 \frac{\log(\sin^2 x) - \log x - \log x}{x - \sin x}$$

20) $$\lim_{x \to 0^+} \frac{(1+x^3)^{\log x} - 1}{\log x \, \log(1 + x - \sin x)}$$