Theorem: $f: D \to \mathbb{R}$
$x_0 \in Acc(D)$. Suppose $\exists I$ neighborhood of x_0 s.t.

1) $f(x) = g(h(x)) \quad \forall x \in I \setminus \{x_0\}$
2) $\lim_{x \to x_0} h(x) = y_0$
3) $\boxed{h(x) \neq y_0} \quad \forall x \in I \setminus \{x_0\}$
4) $\lim_{y \to y_0} g(y) = \ell$

Then
$$\lim_{x \to x_0} f(x) = \ell$$

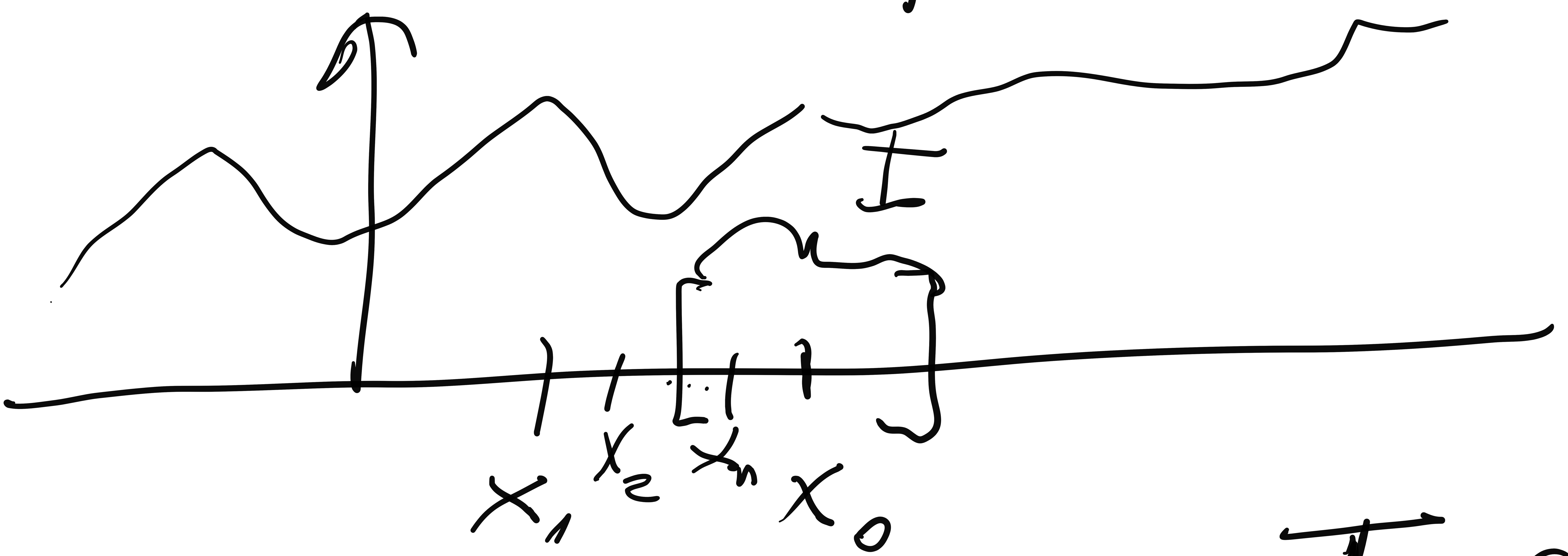
Proof $\lim_{x \to x_0} f(x) = \ell$

$\Updownarrow$

$\forall$ sequence $x_n \to x_0$

then $\lim_{n\to\infty} f(x_n) = \ell$

Let x_n a sequence $x_n \to x_0$



$\forall n > N \quad x_n \in I \setminus \{x_0\}$

$$f(x_n) = g(h(x_n)) = g(y_n)$$

$$y_n = h(x_n) \to y_0$$

and $y_n \neq y_0$

Since $\lim g(y) = \ell$

$$\boxed{\lim_{n\to\infty} g(y_n) = \ell}$$

$$\Longrightarrow \quad f(x_n) = g(y_n) \to \ell$$

Since (x_n) is arbitrary the theorem is proved

q.e.d.

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = e^5$$

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n \underset{y=\frac{n}{5}}{=} \lim_{y\to+\infty}\left(\left(1+\frac{1}{y}\right)^y\right)^5$$

$n = 5y$

$$\underset{z=\left(1+\frac{1}{y}\right)^y \to e}{=} \lim_{z\to e} z^5 = e^5$$

A little abuse:

$$\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y = e$$

We had proved only

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$n \in \mathbb{N}$

- a^x is an infinite greater than x^α (for $x \to +\infty$)

$\forall a > 1 \quad \alpha > 0$

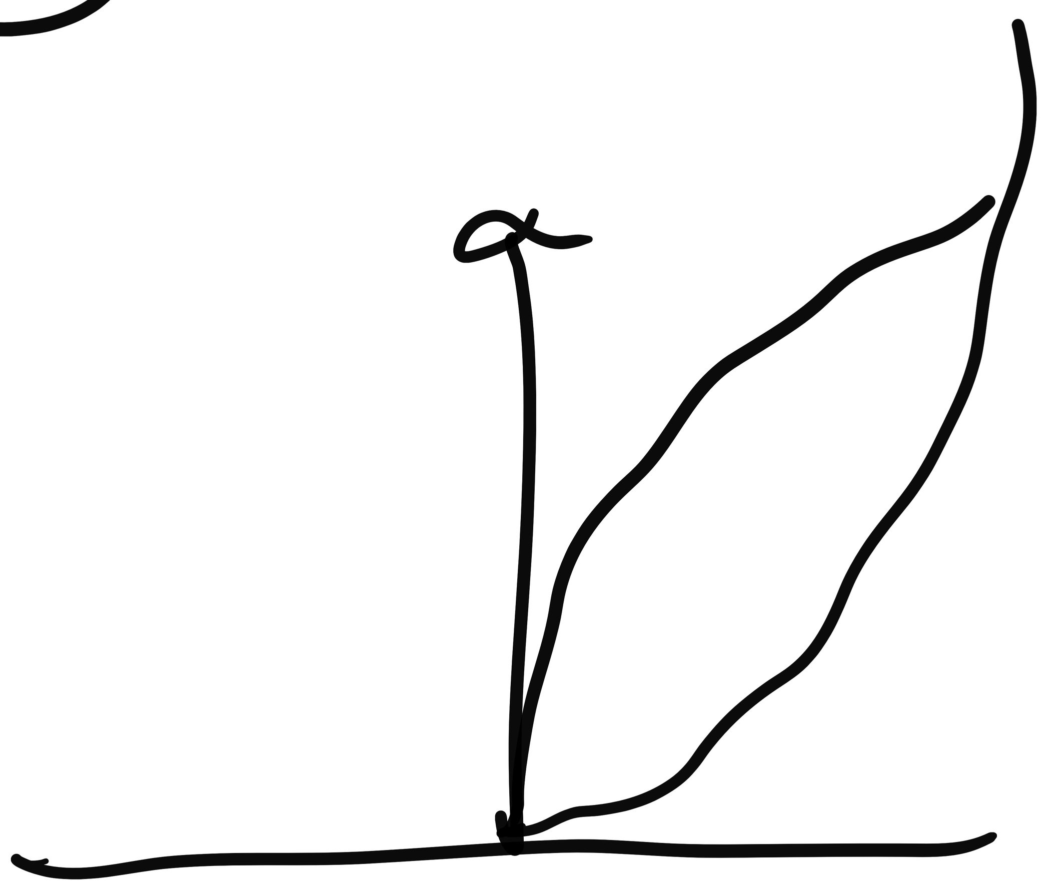
$$\lim \frac{a^x}{x^\alpha} = +\infty$$

- x^α is infinite greater than $\log_b x$

$b > 1$

Exercise $\alpha, \beta > 0$

$$\lim_{x \to 0^+} x^\alpha |\log x|^\beta =$$



$$= \text{"} 0 \cdot +\infty \text{"}$$

$\uparrow$

indeterminate form

$$\lim_{x \to 0^+} x^\alpha |\log x|^\beta \underset{\substack{y = \log x \\ x = e^y}}{=} \lim_{y \to -\infty} e^{\alpha y} |y|^\beta =$$

$$\underset{z = -y}{=} \lim_{z \to +\infty} e^{-\alpha z} |z|^\beta =$$

$$= \lim_{z \to +\infty} \frac{|z|^\beta}{e^{\alpha z}} \underset{\alpha z = w}{=}$$

$$= \lim_{w \to +\infty} \frac{\left|\frac{w}{\alpha}\right|^\beta}{e^w} = \frac{1}{\alpha^\beta} \lim_{w \to +\infty} \frac{|w|^\beta}{e^w}$$

$= 0.$

So we have proved

$$\lim_{x \to +\infty} x^{\alpha} |\log x|^{\beta} = 0$$

$\forall \alpha, \beta > 0$

<u>Exercise</u>:

$$\lim_{x \to 0+} \left(\frac{1}{x}\right)^{x}$$

it is a form $+\infty^{0}$

$$\lim_{x \to 0+} \left(\frac{1}{x}\right)^{x} = \lim_{x \to 0+} e^{\log\left(\left(\frac{1}{x}\right)^{x}\right)} =$$

$$= \lim_{x \to 0+} e^{x \log\left(\frac{1}{x}\right)} =$$

$$= \lim_{x \to 0^+} e^{x(\log 1 - \log x)} =$$

$$= \lim_{x \to 0^+} e^{-x \log x} =$$

$$\lim_{z \to 0} e^{-z} = 1$$

$z = -x \log x \quad z \to 0$

$$\lim_{x \to 0^+} x \log x = 0$$

Proposition: starting $\left(1+\frac{1}{n}\right)^n \to e$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \text{I}$$

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \text{II}$$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e \qquad \text{III}$$

Proof

II

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} \underset{y=-x}{=}$$

$$\lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y} =$$

$$\lim_{y\to+\infty}\left(\frac{y-1}{y}\right)^{-y} =$$

$$\lim_{y\to+\infty}\left(\frac{y-1+1}{y-1}\right)^{y} =$$

$$\lim_{y\to+\infty}\left[\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)\right]$$

$$= \lim_{z\to+\infty}\underbrace{\left(1+\frac{1}{z}\right)^{z}}_{\to e}\underbrace{\left(1+\frac{1}{z}\right)}_{\to 1} = e$$

$z=y-1$

$$\boxed{\lim_{x\to 0+}}(1+x)^{\frac{1}{x}} \underset{y=\frac{1}{x}}{=} \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}=e$$

$$\boxed{\lim_{x\to 0-}}(1+x)^{\frac{1}{x}} \underset{y=\frac{1}{x}}{=} \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{y}=e$$

$$\Longrightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$$

$$\left(1+\frac{1}{[x]+1}\right)^{\cancel{x}} \le \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{[x]}\right)^{x}$$

$$\left(1+\frac{1}{[x]+1}\right)^{x} \ge \left(1+\frac{1}{[x]+1}\right)^{[x]}$$

$$\left(1+\frac{1}{[x]}\right)^{x} \le \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} = \left(1+\frac{1}{[x]+1}\right)^{[x]+1}\left(1+\frac{1}{[x]+1}\right)^{-1}$$

$$\left(1+\frac{1}{[x]}\right)^{[x]+1} = \left(1+\frac{1}{[x]}\right)^{[x]}\cdot\left(1+\frac{1}{[x]}\right)$$

$$\underbrace{\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\left(1+\frac{1}{[x]+1}\right)^{-1}}_{a} \leq \left(1+\frac{1}{x}\right)^{x} \leq \underbrace{\left(1+\frac{1}{[x]}\right)^{[x]}\left(1+\frac{1}{[x]}\right)}_{b}$$

$$a: \underbrace{\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1}}\ \underbrace{\left(1+\frac{1}{[x]+1}\right)^{-1}}_{\to 1} =$$

$$= \lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1} \underset{n=[x]+1}{=}$$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

$[x]$ integer part of $x > 0$

$$[x] := \max \{ n \in \mathbb{N} \text{ s.t. } n \leq x \}$$

$x = 1,9$ $\quad [x] = 1$

$x = 27,102$ $\quad [x] = 27$

<u>Corollary</u>:

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{(1+x)^a - 1}{x} = a \quad \forall a > 0$$

Proof

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \log(1+x) =$$

$$= \lim_{x \to 0} \log \boxed{(1+x)^{\frac{1}{x}}} =$$

$$\underset{z = (1+x)^{\frac{1}{x}}}{=} \lim_{z \to e} \log z = \log e = \underline{1}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} \underset{y = e^x - 1}{=} \lim_{y \to 0} \frac{y}{\log(1+y)}$$

$$e^x = y + 1$$

$$x = \log(y+1)$$

$$= \lim_{y \to 0} \frac{1}{\underbrace{\frac{\log(1+y)}{y}}_{\to 1}} = \underline{1}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad x \to 0$$

$$\boxed{\sin x = x + o(x)}$$

$$\frac{\sin x}{x} = \frac{\cancel{x}}{\cancel{x}} + \frac{o(x)}{x}$$

$$\downarrow x \to 0$$

$$\lim \frac{\sin x}{x} = 1 + \lim \frac{o(x)}{x} = 1$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) \qquad x \to 0$$

$$\Updownarrow$$

$$\frac{\cos x - 1}{x^2} = -\frac{1}{2} + \frac{o(x^2)}{x^2}$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

$$\lim_{x \to 0} \frac{(\cos x - 1)}{x^2} \frac{(\cos x + 1)}{(\cos x + 1)} =$$

$$= \lim_{x \to 0} \frac{\cos^2 x - 1}{x^2} \cdot \frac{1}{(\cos x + 1)} =$$

$$= \lim_{x \to 0} - \left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{(\cos x + 1)}$$

$$= -1 \cdot \frac{1}{2} = -\frac{1}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

hyperbolic sine

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$(\cosh(x))^2 - (\sinh(x))^2 = \underline{1}$$