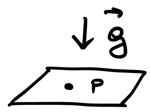
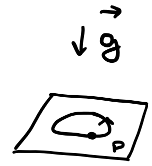
Lesson 15 - 03/11/2022

- Dynamics on a constrained dynamical system on a surface (dim 2) on a curve (dim 1): $\vec{\phi}$ are unknows !!



- EXAMPLES:
  - Fixed point on a table
  - Point in uniform circular motion on a table



    ($\vec{\phi}$ may depend on velocities !!)
  - Pendulum ($\vec{\phi}$ are , in general , difficult to calculate !!)

- NOTION of IDEAL CONSTRAINT , consequences on Newton's eqs.
- Kinetic energy in terms of $\vec{q}$ for a system of N points of a constrained dyn syst.

—x—x—

→ Surface S (dim = 2)

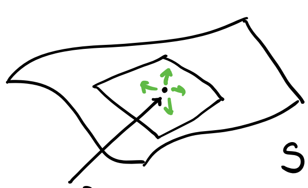
↗ as 0 of a function $F(x,y,z)=0$ $F \in C^\infty$ such that $\nabla F(x,y,z)|_S \neq 0$

↘ by a local parameterization with 2-parameters.

$$S \ni \vec{w} = (x,y,z) = (x(q_1,q_2),\, y(q_1,q_2),\, z(q_1,q_2))$$

$$\frac{d\vec{w}}{d\vec{q}} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} \end{pmatrix} \text{ has rank } = 2.$$

$$\underset{\displaystyle \frac{\partial \vec{w}}{\partial q_1}}{\uparrow\,=} \qquad \underset{\displaystyle \frac{\partial \vec{w}}{\partial q_2}}{\uparrow\,=}$$



S

$S \ni \vec{w}$

$$\underset{T_{\vec{w}} S}{\underset{\cap}{\delta \vec{w}}} = \sum_{h=1}^{2} \frac{\partial \vec{w}}{\partial q_h} \delta q_h$$

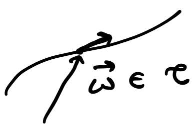
VIRTUAL DISPACEMENT.

→ Curve $\mathcal{C}$ (dim = 1) ↗ as a 0 of a system of functions $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ s.t. $rk \begin{pmatrix} \nabla F \\ \nabla G \end{pmatrix}$ is max $= 2$

$F, G \in C^\infty$.

$$\mathcal{C} \ni \vec{w} = (x, y, z) = (x(q_1), y(q_1), z(q_1))$$

$$\frac{d\vec{w}}{dq_1} = \begin{pmatrix} \partial x/\partial q_1 \\ \partial y/\partial q_1 \\ \partial z/\partial q_1 \end{pmatrix} \neq 0.$$



$$\underset{\Uparrow}{\delta\vec{w}} = \frac{\partial \vec{w}}{\partial q_1}\,\delta q_1$$

$T_{\vec{w}}\,\mathcal{C}$

Dynamics on a constrained system ?

- In the unconstrained case, Newton's eqs don't present problems since:

$$m\vec{a} = \vec{F}$$

$$\begin{cases} m\ddot{x} = F_x(\vec{OP}, \vec{v}, t) = \vec{F}\cdot\vec{e}_1 \\ m\ddot{y} = F_y(\vec{OP}, \vec{v}, t) = \vec{F}\cdot\vec{e}_2 \\ m\ddot{z} = F_z(\vec{OP}, \vec{v}, t) = \vec{F}\cdot\vec{e}_3 \end{cases}$$

THE PROBLEM IS WELL-POSED : 3 eqs AND 3 unknows.
↙
x, y, z.

- In the constrained case, Newton's eqs becomes :

$$m\vec{a} = \vec{F} + \vec{\phi} \qquad \vec{\phi} = (\phi_1, \phi_2, \phi_3) = \vec{\phi}(\vec{OP}, \vec{v}, t)$$

THE PROBLEM IS NOT WELL-POSED :

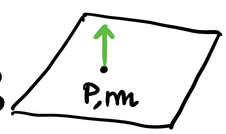
3 eqs AND $h = 1, 2$ + 3 unknows for $\vec{\phi}$ → 4 OR 5 UNKNOWNS

$h = 1$ ↙ curve, $h = 2$ ↘ surface

$\downarrow \vec{g}$

Examples

- Fixed point on a table.



$\uparrow \vec{e}_3$

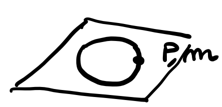
$$m\vec{a} = 0 = \vec{F} + \vec{\phi} = -mg\vec{e}_3 + \vec{\phi}$$

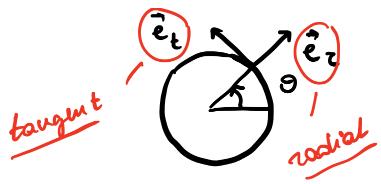
↓ P is fixed

$\Downarrow$ $\vec{\phi} = mg\vec{e}_3$

$\downarrow \vec{g}$

- Point in uniform circular motion on a table.





$$\vec{a} = \underbrace{R\ddot{\theta}}\,\vec{e}_t - \frac{(R\dot{\theta})^2}{R}\vec{e}_r$$

$$\left(\vec{a} = \ddot{s}\,\vec{e}_t - \frac{\dot{s}^2}{\rho}\vec{e}_r \right)$$

$s = R\theta$ for the circle

Our motion is uniform : $\ddot{\theta} = 0$. Therefore :

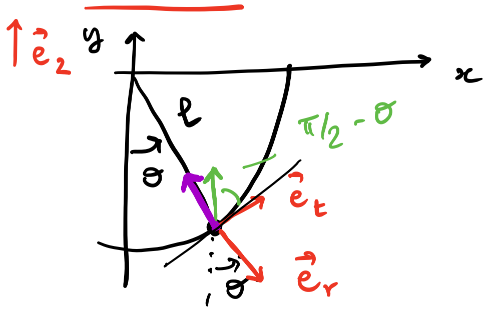
$$\vec{a} = 0\,\vec{e}_t - \frac{R^2\dot{\theta}^2}{R}\vec{e}_r = -R\dot{\theta}^2\vec{e}_r$$

Hence :

$$m\vec{a} = -mR\dot{\theta}^2\vec{e}_r = \vec{F} + \vec{\phi} = -mg\vec{e}_3 + \vec{\phi}$$

$$\boxed{\vec{\phi} = mg\vec{e}_3 - mR\dot{\theta}^2\vec{e}_r}$$

- Pendulum



P, m, ℓ = lenght

θ = "Lagrangian" coo.

$$\boxed{m\ell\ddot{\theta} = -mg\sin\theta}$$

eqs for the pendulum of mass m and lenght ℓ.

$\Downarrow$

$$V(\theta) = -mg\ell\cos\theta$$

$\vec{\phi} = \vec{\phi}(\theta)$ for a motion $(\theta_0, v_0) = (\frac{\pi}{2}, 0)$

$$\vec{a} = \ddot{s}\,\vec{e}_t - \frac{\dot{s}^2}{\ell}\vec{e}_r = \ell\ddot{\theta}\,\vec{e}_t - \frac{\ell^2\dot{\theta}^2}{\ell}\vec{e}_r =$$

$$= \ell\ddot{\theta}\,\vec{e}_t - \ell\dot{\theta}^2\vec{e}_r.$$

conservation of energy.

we use eq of motion

$$m\vec{a} = \vec{F} + \vec{\phi} \Rightarrow \vec{\phi} = m\vec{a} - \vec{F}$$

gravit. force

$\ddot{\theta}$: use the eq. of motion.

$\cancel{m}\ell\ddot{\theta} = -\cancel{m}g\sin\theta \Rightarrow \ddot{\theta} = -\dfrac{g\sin\theta}{\ell}$

$\dot{\theta}^2$: use the conservation of energy.

$\frac{1}{2}\cancel{m}(\ell^{\cancel{2}}\dot{\theta}^2) - \cancel{m}g\cancel{\ell}\cos\theta = 0 \Rightarrow \dot{\theta}^2 = \dfrac{2g\cos\theta}{\ell}$

$\downarrow$

total energy at time 0. $(\theta_0 = \frac{\pi}{2},\ \dot{\theta}_0 = v_0 = 0)$

$$\vec{\phi} = -\underbrace{\vec{F}}_{+mg\vec{e}_2} + m\vec{a} = mg\vec{e}_2 + m\cancel{\ell}\left(\frac{-g\sin\theta}{\cancel{\ell}}\right)\vec{e}_t - m\cancel{\ell}\frac{2g\cos\theta}{\cancel{\ell}}\vec{e}_r =$$

$$= \boxed{mg\vec{e}_2 - mg\sin\theta\,\vec{e}_t - 2mg\cos\theta\,\vec{e}_r}$$

write $\vec{e}_2$ in terms of $\vec{e}_t$ and $\vec{e}_r$

$\vec{e}_2 = \sin\theta\,\vec{e}_t - \cos\theta\,\vec{e}_r$

Finally

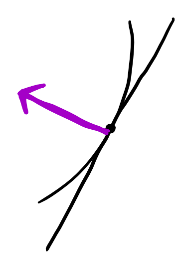
$$\vec{\phi} = \cancel{-mg\sin\theta\,\vec{e}_t} - 2mg\cos\theta\,\vec{e}_r + \cancel{mg\sin\theta\,\vec{e}_t} - mg\cos\theta\,\vec{e}_r$$

$$= -3mg\cos\theta\,\vec{e}_r$$

$\to$ Notice that, at every point of the constraint, $\vec{\phi} \perp T_p S^1$

$\downarrow$

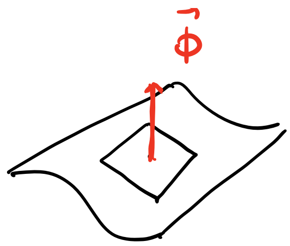
This means that the constraint is "IDEAL".

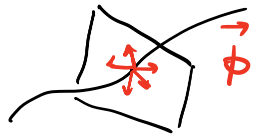


Definition - IDEAL CONSTRAINT -

The constraint $\vec{\phi}$ is called ideal if $\boxed{\vec{\phi} \perp T_{\vec{\omega}} Q}$

(where Q has dim 2, 1).





iff $\vec{\phi} \cdot \delta\vec{\omega} = 0 \quad \forall\ \delta\vec{\omega} \in T_{\vec{\omega}} Q.$

iff $\left(\delta\vec{\omega} = \sum_h \frac{\partial \vec{\omega}}{\partial q_h} \delta q_h \right) \quad \vec{\phi} \cdot \frac{\partial \vec{\omega}}{\partial q_h} = 0 \quad \forall h.$ $(h = 1 \text{ or } 2).$

Now we can return to Newton's eqs. for a constrained dyn. system.

$$m\vec{a} = \vec{F} + \vec{\phi} \implies (m\vec{a} - \vec{F}) \cdot \frac{\partial \vec{\omega}}{\partial q_h} = \underbrace{\vec{\phi} \cdot \frac{\partial \vec{\omega}}{\partial q_h}}_{=0}$$

h eqs. AND h unknowns

↓

<u>THE PROBLEM IS WELL-POSED !!</u>

$\boxed{\text{kinetic energy for a constrained dynamical system.}}$

Consider $P_1 \dots P_N$ points of masses m_1 — m_N

$K_i = \frac{1}{2} m_i |\underset{\overset{\parallel}{\dot{\vec{OP_i}}}}{\vec{v}_i}|^2$ the kinetic energy of the point P_i of mass m_i.

If the point is constrained on a mnf (dim 1 or 2).

$\vec{OP_i} = \vec{OP_i}(\vec{q}, t)$

$$\vec{v}_i = \dot{\vec{OP_i}}(\vec{q}, \dot{\vec{q}}, t) = \sum_{h=1}^{m} \frac{\partial \vec{OP_i}}{\partial q_h}(\vec{q}, t)\, \dot{q}_h + \frac{\partial \vec{OP_i}}{\partial t}(\vec{q}, t)$$

<u>On monday.</u>

$$K = \frac{1}{2} \sum_{i=1}^{N} m_i |\vec{v}_i|^2 =$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \left[\ . \ . \ . \ . \ \right]$$

—x—x—