$\frac{\infty}{\infty}$ $\frac{0}{0}$ indeterminate forms

$\infty \cdot 0$

$1^{\pm\infty}$, $(0_+)^0$, $(0_+)^{+\infty}$

$$\lim_{x \to x_0} f(x)^{g(x)} = e^{\log\left(f(x)^{g(x)}\right)}$$

$$\lim_{x \to x_0} = e^{\underbrace{g(x) \cdot \log(f(x))}_{\to l}}$$

$$= e^{l}$$

$\lim f(x) \to 1$

$\lim g(x) \to +\infty$

$$\underbrace{g(x)}_{+\infty} \underbrace{\log(f(x))}_{\to 0}$$

Def f is infinitesimal at $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ $\lim_{x \to x_0} f(x) = 0$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to +\infty} \frac{1}{2x} = 0$$

$$\lim_{x \to +\infty} \frac{1}{2x + x^3} = 0$$

$$\lim_{x \to +\infty} \frac{5}{\sqrt{x}} = 0$$

Def Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$

We say g is an infinitesimal greater (or higher) than f

if $\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0$

$f(x) = \frac{1}{x}$ $\quad g(x) = \frac{1}{2x}$ $\quad x_0 \to +\infty$

$\lim \frac{g(x)}{f(x)} = \frac{1}{2} \neq 0$

$g(x)$ and $f(x)$ are "of the same order"

$f(x) = \frac{1}{2x}$ $\quad g(x) = \frac{1}{2x + x^3}$

they infinitesima for $x \to +\infty$

$$\lim_{x \to \infty} \frac{g(x)}{h(x)} = \frac{2x}{2x + x^3} =$$

$$= \frac{2}{2 + x^2} = 0$$

$$f(x) = \sqrt[3]{x} \quad g(x) = \sqrt[2]{x}$$

$\boxed{x \to 0^+}$ infinitesimi

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim \frac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}} =$$

$$\lim x^{\frac{1}{3} - \frac{1}{2}} =$$

$$\lim_{x \to 0^+} x^{-\frac{1}{6}} = \frac{1}{x^{\frac{1}{6}}} = +\infty$$

$$\lim_{x \to 0} \frac{g(x)}{f(x)} = \lim_{x \to 0} x^{\frac{1}{6}} = 0$$

Compare infinites:

$$\lim_{x \to +\infty} \frac{g(x)}{f(x)} = +\infty$$

$$\Updownarrow$$

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0$$

$$\lim_{x \to 0^+} \frac{x^2 + x^3}{\sqrt{x} + x} =$$

$$= \lim_{x \to 0^+} \frac{x^2(1+x)}{\sqrt{x}(1+\sqrt{x})} =$$

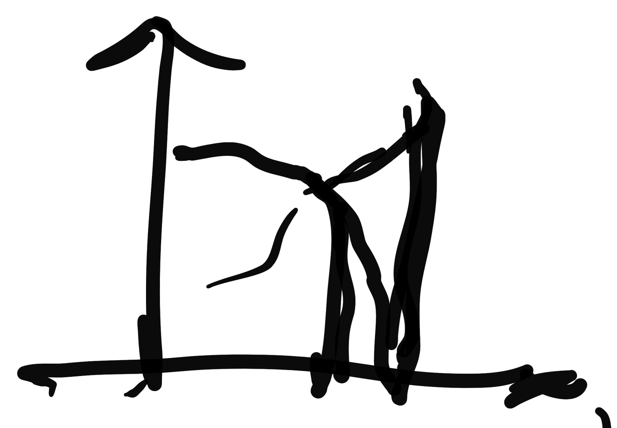
$$\to 1 \qquad \to 1$$

$$= \lim_{x \to 0^+} x^{\frac{3}{2}} = 0$$

$$\lim_{x\to+\infty}\frac{x^2+x^3}{\sqrt{x}+x}=\frac{x^3\overbrace{\left(\frac{1}{x}+1\right)}^{\to 1}}{x\underbrace{\left(\frac{1}{\sqrt{x}}+1\right)}_{\to 1}}$$

$$=\lim_{x\to+\infty} x^2\cdot\Big(\quad\Big)^{\to 1} = +\infty$$

$$\boxed{\lim_{y\to 0}\frac{\sin y}{y}=1}$$



$\sin y \le y \le \operatorname{tg} y$

(I) $$\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}\overset{?}{=}1$$

(II) $$\lim_{x\to 2}\frac{\sin(\overbrace{x^2-4}^{\to 0})}{\underbrace{x^2-4}_{\to 0}}\overset{?}{=}1$$

Structure:

in (I) $f(x)=\dfrac{\sin(\sin x)}{\sin x}=g(h(x))$

$$g(y)=\frac{\sin y}{y}$$

$h(x)=\sin x$

Theorem "change of variable"

Theorem $f : D \to \mathbb{R}$, x_0 acc point for D. Suppose $\exists I$ neigh. of x_0

1) $f(x) = g(h(x)) = g \circ h(x)$ $\quad \forall x \in I \setminus \{x_0\}$

2) $\boxed{\lim_{y \to y_0} g(y) = l} \in \mathbb{R} \cup \{\pm\infty\}$

3) $\lim_{x \to x_0} h(x) = y_0$

4) $h(x) \neq y_0$ $\quad \forall x \in I \setminus \{x_0\}$

Then $\boxed{\lim_{x \to x_0} f(x) = l}$

Example $\lim_{x \to \infty} \dfrac{e^{\sqrt{\log x}}}{\sqrt{x}}$

horrible: $e^{\sqrt{\log x}} = \sqrt{e^{\log x}} = \sqrt{x}$ NO !!!!

$g(y) = \boxed{\dfrac{e^{y}}{\sqrt{e^{y^2}}}}$

$y = \sqrt{\log x} = \boxed{h(x)}$ $\quad y^2 = \boxed{\log x}$

$\lim_{x \to x_0 = +\infty} h(x) = y_0 = +\infty$ $\quad x = e^{y^2}$ $\quad \sqrt{x} = \sqrt{e^{y^2}}$

$$\boxed{\lim_{x \to +\infty} \frac{e^{\sqrt{\log x}}}{\sqrt{x}}} \underset{y = \sqrt{\log x}}{=} \lim_{y \to +\infty} \frac{e^{y}}{\sqrt{e^{y^2}}} =$$

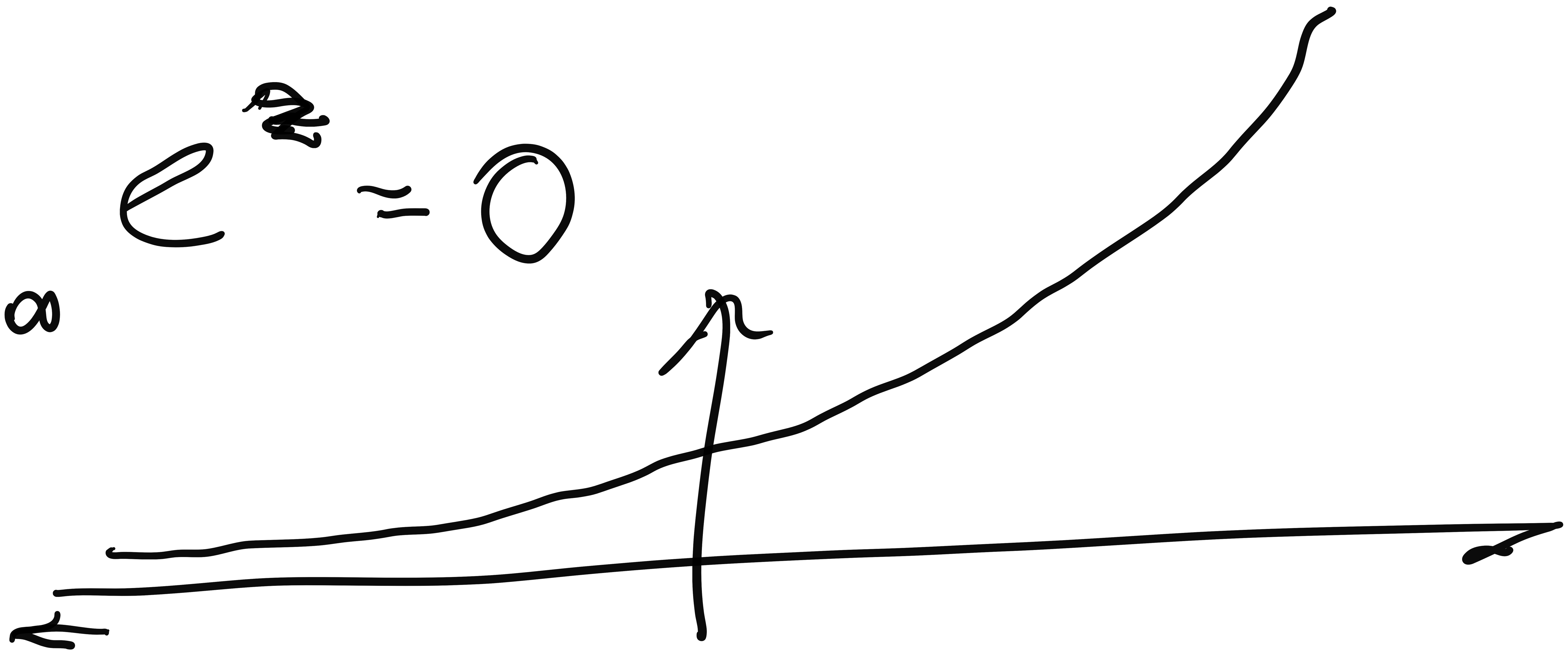
$$\lim_{y\to+\infty} \frac{e^y}{\sqrt{e^{y^2}}} = \lim_{y\to+\infty}\frac{e^y}{e^{\frac{y^2}{2}}} = \lim_{y\to+\infty} e^{y-\frac{y^2}{2}}$$

$$= 0$$

$$\lim_{y\to+\infty} y - \frac{y^2}{2} = \lim_{y\to+\infty} \overset{\to+\infty}{y^2}\overset{\to-\frac{1}{2}}{\left(\frac{1}{y}-\frac{1}{2}\right)} = -\infty$$

$$e^{y-\frac{y^2}{2}} \xrightarrow{y\to+\infty} 0$$

$$\lim_{z\to-\infty} e^z = 0$$



$$\lim_{x\to+\infty} \left(\frac{1}{\log x}\right)^{\frac{\log x}{x}} =$$

$$0$$

$$\lim_{x\to a} \boxed{f(x)^{g(x)}}$$

$$0 \leftarrow f(x) = \frac{1}{\log x} \qquad g(x) = \frac{\log x}{x}$$

$$\lim_{x\to+\infty} e^{\log\left[\left(\frac{1}{\log x}\right)^{\frac{\log x}{x}}\right]} =$$

$0^0 \leftarrow$ indeterminato

$$\lim_{x\to+\infty} e^{\frac{\log x}{x}\log\left(\frac{1}{\log x}\right)}$$

$$\lim_{x\to+\infty} e^{\frac{\log x}{x}\left(\log 1 - \log(\log x)\right)}$$

$$\lim_{x\to+\infty} e^{-\frac{\log x}{x}\cdot\log(\log x)}$$

$$\lim_{x\to+\infty} \frac{\log x\cdot\log(\log x)}{x} =$$

$y = \log x$

$x = e^y$

$$\lim_{y\to+\infty} \frac{y\cdot\log y}{e^y} =$$

$$= \lim_{y \to +\infty} \underbrace{\frac{\log y}{y}}_{0} \cdot \underbrace{\frac{y^2}{e^y}}_{0} = 0$$

$$\Rightarrow \boxed{\lim_{x \to +\infty} \left(\frac{1}{\log x}\right)^{\frac{\log x}{x}} = e^{\lim(\text{exponent})}}$$

$$= e^{0} = 1.$$

(We have used $z \to e^z$ is continuous)

The notation

$\boxed{g = o(h)}$ for $x \mapsto x_0$

means a function g such that $\boxed{\lim_{x \to x_0} \left(\frac{g}{h}\right) = 0}$

$$\boxed{\sin x = x + o(x) \quad x_0 = 0}$$

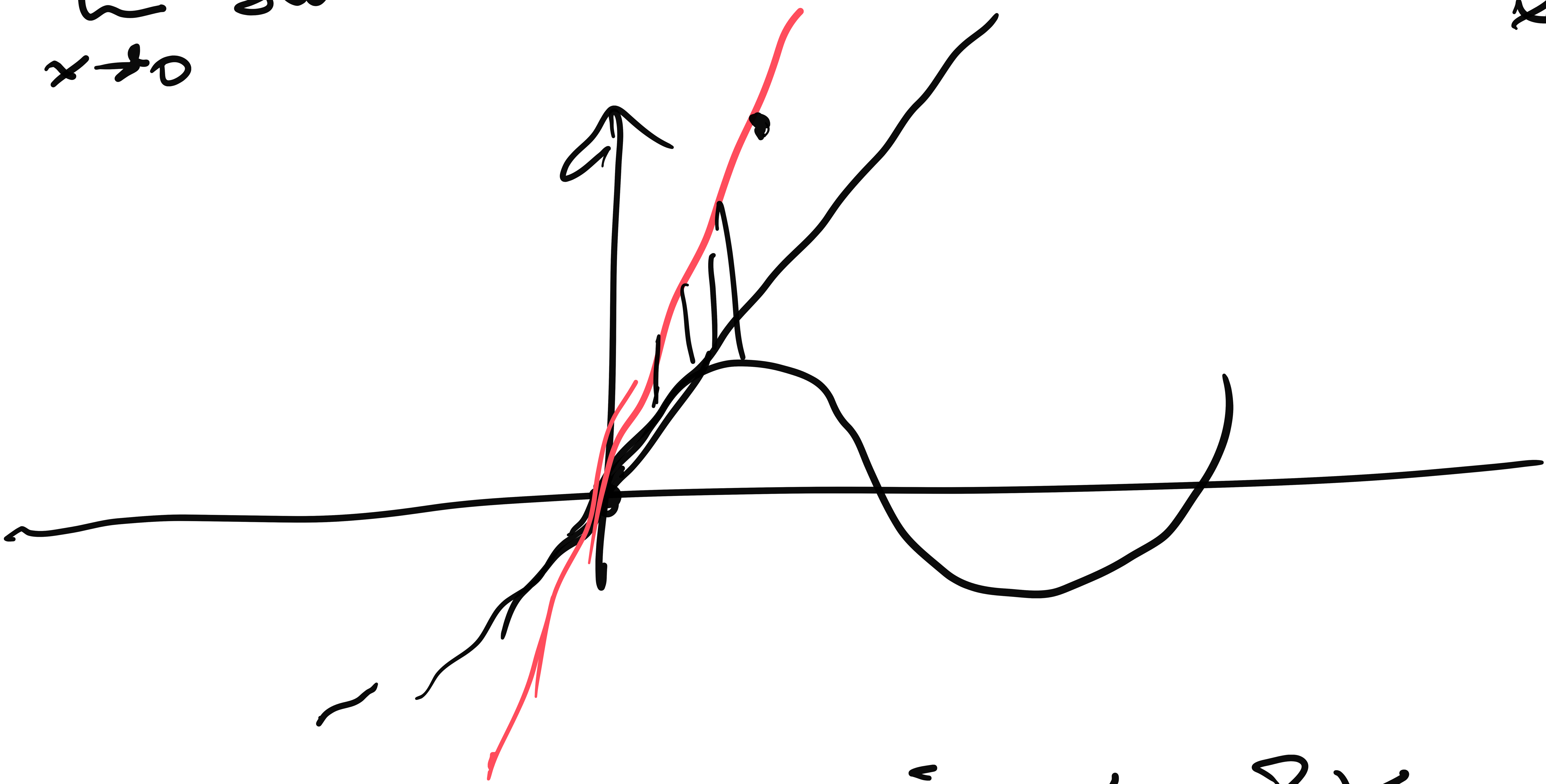
$$\boxed{\sin x = X + g(x)}$$ and

$$\lim_{x \to 0} \frac{g(x)}{x} = 0$$

We already know

$$\lim_{x\to 0} \sin x - x = 0$$

$$\lim \frac{\sin x - x}{x} \to 0$$



$y = 2x$

$$\frac{\sin x - 2x}{x} \not\to 0$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\parallel$$

$$g$$

$x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + g \quad \lim \frac{g}{x^2} \to 0$$