# Chapter 2

# Lagrangian Mechanics

## 2.1 Introduction

There are two problems that naturally lead to the Lagrangian formalism within Newtonian Mechanics: the problem of writing the equations of motion in an arbitrary system of coordinates, and the problem, strictly connected to it, of eliminating constraint forces, or *constraint reactions*, in constrained systems. Such problems arise alrady for elementary systems with a single point mass, so we shall illustrate them here in such a simple case, deferring the general treatment to the next sections. A further important motivation to be interested in the Lagrangian formalism is its connection with other disciplines, like geometry and optics, or with general problems of optimization, through the so-called variational formulation of the laws of mechanics. To such an important aspect of Lagrangian mechanics we shall devote the last sections of this chapter.

### 2.1.1 The point mass in arbitrary coordinates

The Newton equation for a point $P$ of mass $m$, subject to a force $\mathbf{F}(P, \mathbf{v}, t)$ depending on its position, its velocity and possibly on time, has the well known form

$$m\mathbf{a} = \mathbf{F} \ . \tag{2.1.1}$$

Such a vector equation coresponds to three scalar equations: if $x, y, z$ are the Cartesian coordinates of $P$ in a given reference frame of origin $O$ and axes $x, y, z$, with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, then (2.1.1) is equivqlent to the system

$$m\ddot{x} = F_x(P, \mathbf{v}, t) \ , \qquad m\ddot{y} = F_y(P, \mathbf{v}, t) \ , \qquad m\ddot{z} = F_z(P, \mathbf{v}, t) \ , \tag{2.1.2}$$

where $F_x$, $F_y$ e $F_z$ denote the components of $\mathbf{F}$ in the chosen frame, that is $F_x = \mathbf{F} \cdot \mathbf{e}_x$ and so on. Such equations, as is known, are invariant for a rather wide class of coordinate changes – the so-called Galilean transformations – including translations and rotations of the frame, as well as passing to coordinate systems in uniform rectilinear motion: if we change coordinates within this class, the equations of motion in the new coordinates mantain the elementary form (2.1.2).

The form (2.1.2) of the equations is instead lost if we pass to more general coordinates, like the familiar cylindrical or polar coordinates. Let us restrict ourselves, for simplicity, on the plane, and consider the usual polar coordinates $r, \vartheta$, searching for the equations of motion in such coordinates; the relation between polar and Cartesian coordinates, we recall, is

$$x = r\cos\vartheta \ , \qquad y = r\sin\vartheta \ .$$

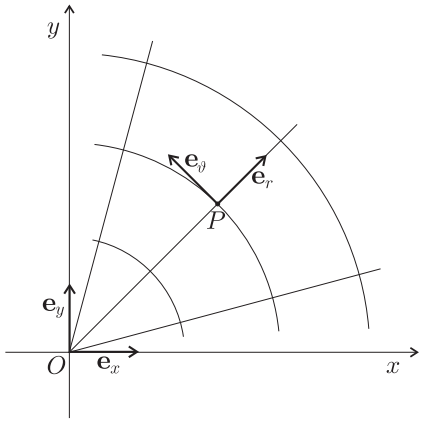


Figure 2.1: *Illustrating the passage from Cartesian to polar coordinates in the plane.*

To deduce from (2.1.1) two scalar equations for $r$ and $\vartheta$, we should project it, this time, not along $\mathbf{e}_x$ e $\mathbf{e}_y$, but along the unit vectors $\mathbf{e}_r$ e $\mathbf{e}_\vartheta$ tangent to the new coordinate lines, named respectively *radial* and *transversal* (figure 2.1). Such vectors, at variance with $\mathbf{e}_x$ e $\mathbf{e}_y$, are different from place to place, and so, for any motion, they change in time. Let us then observe, preliminarly, that

Lemma 16 *It is*

$$\dot{\mathbf{e}}_r = \dot\vartheta \mathbf{e}_\vartheta \ , \qquad \dot{\mathbf{e}}_\vartheta = -\dot\vartheta \mathbf{e}_r \ . \tag{2.1.3}$$

Proof. These expressions immediately follow from

$$\mathbf{e}_r = \cos\vartheta \mathbf{e}_x + \sin\vartheta \mathbf{e}_y \ , \qquad \mathbf{e}_\vartheta = -\sin\vartheta \mathbf{e}_x + \cos\vartheta \mathbf{e}_y \ ,$$

taking a time derivative. □

It is now easy to proceed: if $P - O = r\mathbf{e}_r$ is the vector giving the position of our point $P$, by deriving with respect to time and using the (2.1.3), we first find the velocity of $P$, namely $\mathbf{v} = \dot r \mathbf{e}_r + r\dot\vartheta \mathbf{e}_\vartheta$, and then its acceleration

$$\mathbf{a} = (\ddot r - r\dot\vartheta^2)\, \mathbf{e}_r + (r\ddot\vartheta + 2\dot r\dot\vartheta)\, \mathbf{e}_\vartheta \ .$$

The acceleration appears now decomposed into its radial and transversal components, namely

$$a_r = \ddot r - r\dot\vartheta^2 \ , \qquad a_\vartheta = r\ddot\vartheta + 2\dot r\dot\vartheta \ .$$

We can finally write the equations of motion $ma_r = F_r$, and $ma_\vartheta = F_\vartheta$, where $F_r = \mathbf{F}\cdot\mathbf{e}_r$ and $F_\vartheta = \mathbf{F}\cdot\mathbf{e}_\vartheta$ denote, respectively, the radial and transversal components of the force. As a result, the equations of motion assume the form, *different from (2.1.2),*

$$m\ddot r = F_r + mr\dot\vartheta^2 \ , \qquad mr\ddot\vartheta = F_\vartheta - 2m\dot r\dot\vartheta \ .$$

The natural question arises whether there exists a general form of the equations of motion, more general than (2.1.2), which remains invariant under *any* change of coordinates. The question can be formulated, in general terms, as follows: denote $\mathbf{w} = (w_1, w_2, w_3) = (x, y, z)$; quite generally, a

change of coordinates from $x, y, z$ to any coordinates $\mathbf{q} = (q_1, q_2, q_3)$ — a traditional notation in Lagrangian mechanics — is a regular map[1]

$$\mathbf{w} = \mathbf{w}(\mathbf{q}) \ ,$$

or equivalentently

$$x = x(q_1, q_2, q_3) \ , \qquad y = y(q_1, q_2, q_3) \ , \qquad z = z(q_1, q_2, q_3) \ ,$$

from a convenient open set $U \subset \mathbb{R}^3$ (the domain of $\mathbf{q}$) to its image $\mathbf{w}(U) \in \mathbb{R}^3$ (the domain of $\mathbf{w}$), such that the Jacobian matrix

$$\frac{\partial \mathbf{w}}{\partial \mathbf{q}} = \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{pmatrix}$$

has rank 3, that is has nonvanishing determinant; such a condition guarantees, at least locally, that the change of coordinates is a diffeomorphism, i.e. a regular invertible map with regular inverse (as is necessary to say the map is a change of coordinates). As is known, the condition on the determinant implies that the three vectors corresponding to the three columns of the matrix, namely

$$\frac{\partial \mathbf{w}}{\partial q_h} = \frac{\partial x}{\partial q_h}\mathbf{e}_x + \frac{\partial y}{\partial q_h}\mathbf{e}_y + \frac{\partial z}{\partial q_h}\mathbf{e}_z \ , \qquad h = 1, 2, 3 \ ,$$

are linearly independent, and so provide a basis in $\mathbb{R}^3$. Each vector $\frac{\partial \mathbf{w}}{\partial q_h}$ turns out to be tangent[2] to the $h$–th *coordinate line,* namely the line obtained by letting only $q_h$ to vary; figure 2.2 shows the situation for the common spherical polar coordinates $(q_1, q_2, q_3) = (\rho, \vartheta, \varphi)$ defined by

$$x = \rho \sin\vartheta \cos\varphi \ , \qquad y = \rho \sin\vartheta \sin\varphi \ , \qquad z = \rho \cos\vartheta \ . \tag{2.1.4}$$

In the very particular case of Cartesian coordinates, it is $\mathbf{w} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$; the coordinate lines are then straight lines parallel to the coordinate axes, and the Jacobian matrix is the identity.

Exercise 25 *For the above defined spherical polar coordinates, write the three vectors* $\frac{\partial \mathbf{w}}{\partial \rho}$, $\frac{\partial \mathbf{w}}{\partial \vartheta}$, $\frac{\partial \mathbf{w}}{\partial \varphi}$ *tangent to the coordinate lines, and determine where the Jacobian matrix has rank lower than three (that is, where spherical coordinates are not good).*

It is now clear that the general procedure to obtain from the Newton vector equation three scalar differential equations for the unknowns $q_h(t)$, $h = 1, 2, 3$, is to project the equation, point by point, along the three directions tangent to the coordinate lines, that is to write

$$(m\mathbf{a} - \mathbf{F}) \cdot \frac{\partial \mathbf{w}}{\partial q_h} = 0 \ , \qquad h = 1, 2, 3 \ . \tag{2.1.5}$$

We shall see in the next section that such an expression, properly developed, leads to equations of motion that *no matter how the coordinates have been chosen,* have one and the same form, namely that of the Lagrange equations. Before going on, however, it is worthwhile to observe that equations similar to (2.1.5) naturally appear also in connection with the dynamics of the constrained point mass.

[1] We are here excluding, for simplicity, changes of coordinates which depend explicitly on time.

[2] Think for example of a curve described parametrically by the three functions of time $x(t)$, $y(t)$, $z(t)$. The vector $(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t})$ is the velocity, tangent to the curve. Here, in place of time, we have any of the $q_h$, the remaining ones staying constant.

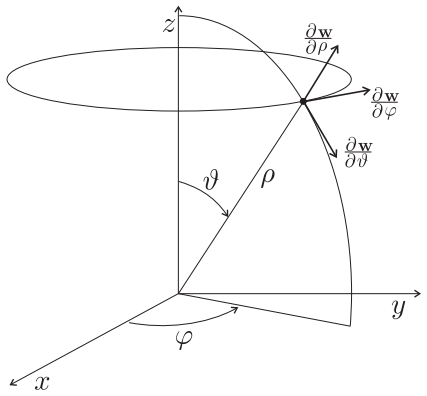


Figure 2.2: *Illustrating the passage from Cartesian to spherical polar coordinates.*

### 2.1.2 The constrained point mass

Let us then come to the problem of the equations of motion for a constrained point mass. We start from kinematics, considering first the case of a point mass constrained to a given susrface $\mathcal{Q}$ (*constraint manifold*, or *configuration space*). We recall that a surface $\mathcal{Q}$ can be defined implicitly, by means of an equation $F(x, y, z) = 0$ with regular $F$ such that[3] its gradient $\nabla F$ never vanishes on $\mathcal{Q}$, or instead in a parametric form. As is well known from mathematical analysis, for each implicitely defined surface, it is always possible to produce locally a parametric representation of the form

$$x = x(q_1, q_2) \ , \qquad y = y(q_1, q_2) \ , \qquad z = z(q_1, q_2) \ , \tag{2.1.6}$$

in compact notation

$$\mathbf{w} = \mathbf{w}(q_1, q_2) \ , \tag{2.1.7}$$

where $(q_1, q_2) \in U \subset \mathbb{R}^2$, $U$ being some open set.[4] In particular, in each of its points, the surface admits a tangent plane, and the $3 \times 2$ Jacobian matrix

$$\frac{\partial(x, y, z)}{\partial(q_1, q_2)} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} \end{pmatrix}$$

has rank 2; this means the two vectors $\frac{\partial \mathbf{w}}{\partial q_h}$, $h = 1, 2$, each being tangent to the coordinate line which is obtained by letting only the $q_h$ coordinate vary, provide, in any point $\mathbf{w} \in \mathcal{Q}$, a basis for the tangent plane $T_{\mathbf{w}}\mathcal{Q}$ to the surface $\mathcal{Q}$ in $\mathbf{w}$. Each tangent vector can then be represented as a linear combination of such vectors (figure 2.3). For tangent vectors, we shall use the traditional notation[5] $\delta\mathbf{w}$, denoting correspondingly by $\delta q_h$, $h = 1, 2$, their components along the basis vectors;

[3] For example, the equation $x^2 + y^2 + z^2 - R^2 = 0$ appropriately defines a surface (the sphere of radius $R$) because $\nabla F = (2x, 2y, 2z)$ never vanishes where $F = 0$. Instead the equation $x^2 + y^2 + z^2 = 0$ does not define a surface but a different object (a point), because $\nabla F$ vanishes where $F = 0$. Which geometrical object remains defined by $x^2 + y^2 - z^2 = 0$ ?

[4] A possible parametric representation (although, in general, not the most convenient one) is obtained by means of the implicit function theorem: if, in a point of the surface, it is, for example, $\frac{\partial F}{\partial z} \neq 0$, then the surface can be described locally as the graph of a function $z = f(x, y)$, with suitable $f$; in such a case the (2.1.6) are given by $x = q_1$, $y = q_2$, $z = f(q_1, q_2)$.

[5] The notation induces to confuse tangent vectors with "small quantities", or with differentials. This is not unpleasent, since the rules to operate on them, starting from (2.1.8), are the same. The correct way to read (2.1.8)

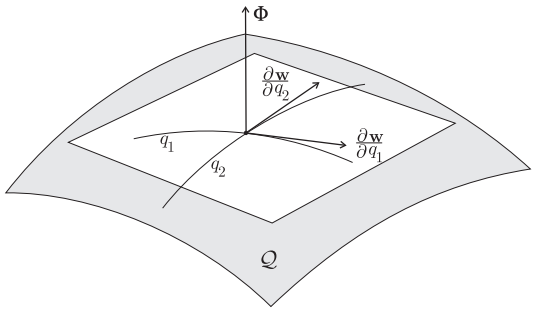


Figure 2.3: *The point mass constrained to a surface.*

that is, we shall write

$$\delta \mathbf{w} = \sum_{h=1}^{2} \frac{\partial \mathbf{w}}{\partial q_h} \delta q_h \ . \tag{2.1.8}$$

The vector $\delta \mathbf{w} \in T_{\mathbf{w}} \mathcal{Q}$ is also called, in the traditional language of mechanics, *virtual displacement* (to be thought of as an *a priori* possible displacement, compatible with the constraint, to be distinguished from the effective displacement which is realized by a given motion passing through $\mathbf{w}$). The local coordinates $q_1$, $q_2$ are called *free coordinates* (to stress the difference with respect to $x, y, z$, which are instead related by $F(x, y, z) = 0$).

Example Consider a spherical pendulum, that is a point mass subject to the constraint $F(x, y, z) = 0$, with $F = x^2 + y^2 + z^2 - R^2$; the condition $\nabla F \neq 0$ on the sphere $F = 0$ is satisfied. In the upper hemisphere $z > 0$ it is possible to use $q_1 = x$ and $q_2 = y$ as free coordinates; the corresponding parametric equations are then

$$x = q_1 \ , \qquad y = q_2 \ , \qquad z = \sqrt{R^2 - q_1^2 - q_2^2} \ , \tag{2.1.9}$$

and the Jacobian matrix $\frac{\partial(x,y,z)}{\partial(q_1,q_2)}$ has everywhere rank 2. In a similar way it is possible to parametrize the lower hemisphere $z < 0$. It is instead necessary to exclude the equator $z = 0$, where equations (2.1.9) are not anymore differentiable. A different much preferable choice of the free coordinates is provided by the angles $\vartheta$ and $\varphi$ of the sperical coordinates; the corresponding parametric equations are

$$x \ = \ R \sin\vartheta \cos\varphi \ , \qquad y \ = \ R \sin\vartheta \sin\varphi \ , \qquad z = R \cos\vartheta \ ,$$

namely the (2.1.4) with $\rho = R$. It is easy to check that the condition on the rank is satisfied everywhere, but on the poles $\vartheta = 0, \pi$.

Let us finally come to the case of a point mass which is constrained to move on a given curve $\mathcal{Q}$. The curve can be defined implicitly through a pair of equations $F(x, y, z) = 0$, $G(x, y, z) = 0$ (figure 2.4), the functions $F$ and $G$ being independent, more precisely such that, on $\mathcal{Q}$, the $2 \times 3$ Jacobian matrix

$$\frac{\partial(F, G)}{\partial(x, y, z)} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{pmatrix}$$

is to look at it as at the tangent (or derivate) application $D\mathbf{w}$ to the application (2.1.7); $D\mathbf{w}$ is indeed a linear application from $T_{\mathbf{q}} U$ to $T_{\mathbf{w}(\mathbf{q})} \mathcal{Q}$, which sends the vector of components $\delta q_1$, $\delta q_2$ into the vector $\delta \mathbf{w}$, and is represented in coordinates by the matrix $\frac{\partial(x,y,z)}{\partial(q_1,q_2)}$.

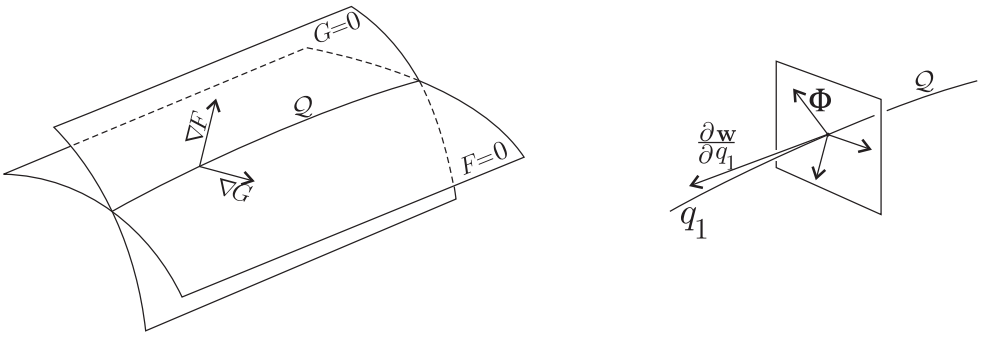


Figure 2.4: *The point mass constrained to a curve.*

has rank 2, and correspondingly, the vectors $\nabla F$, $\nabla G$ are linearly independent;[6] in such a case the position of $P$ on the curve is identified by just one free coordinate, say $q_1$, and the curve admits the parametric representation $x = x(q_1)$, $y = y(q_1)$, $z = z(q_1)$, that is $\mathbf{w} = \mathbf{w}(q_1)$, with the property that $\frac{\partial \mathbf{w}}{\partial q_1} \neq 0$. An easy example is the ordinary plane pendulum of lenght $l$, suspended at the origin of a Cartesian frame $xyz$. If the pendulum is constrained to the vertical plane $xy$, with vertical axis $y$, and $\vartheta$ denotes as usual the angle between the negative $y$ axis and the pendulum, the constraint can be expressed through the pair of equations

$$x^2 + y^2 - l^2 = 0 \ , \qquad z = 0 \ ,$$

while the parametric representation (with $\vartheta = q_1$) is evidently

$$x = l \sin \vartheta \ , \qquad y = -l \cos \vartheta \ , \qquad z = 0 \ .$$

It is then clear that, from a kinematic point of view, the two cases of a point mass constrained to a curve or to a surface are formally similar to the case of the unconstrained point mass described in arbitrary coordinates: in all cases, the position of $P$ is expressed locally (that is, in an open set $U \subset \mathbb{R}^n$) in the parametric form

$$\mathbf{w} = \mathbf{w}(q_1, \ldots, q_n) \ , \tag{2.1.10}$$

by means of three regular functions, with, in the three considered cases, $n = 1$, $n = 2$ or $n = 3$; moreover the vectors $\frac{\partial \mathbf{w}}{\partial q_h}$, $h = 1, \ldots, n$, tangent to the coordinate lines, in every point are linearly independent. The number $n$ is said to be the *number of degrees of freedom* of the system.

Let us now come to the dynamics. While in the case of a free point mass the use of the Newton equation presents no problems, and directly leads to the equations of motion (2.1.5), in the case of a constrained point mass it is necessary to keep into account the action of the physical device which realizes the constraint, that is keeps the point $P$ on the constraint manifold. The first assumption we need to make, is that the device acts by producing a convenient force $\mathbf{\Phi}$, which adds to the other forces. To distinguish, the latter are called the *active forces,* while $\mathbf{\Phi}$ is called the *constraint reaction.* If $\mathbf{F}$ denotes the resultant of the active forces, then the Newton equation writes

$$m\mathbf{a} = \mathbf{F} + \mathbf{\Phi} \ .$$

*While* $\mathbf{F}$ *should be thougt of to be known* (a datum of the problem), $\mathbf{\Phi}$ *instead it is not:* rather, it should be considered as an unknown, which adds to the motion to be determined.

[6] Such a condition replaces the previous condition $\nabla F \neq 0$. It implies that both surfaces $F = 0$, $G = 0$ are well defined, and moreover they intersect transversally (the two vectors $\nabla F$ and $\nabla G$, ortogonal to them, are never parallel; see figure 2.4).

- Consider for example the case of a point $P$ of mass $m$, which is resting on a floor. We know the point is subject to the weight force $\mathbf{F} = -mg\mathbf{e}_z$; if we observe the point does not move, and we admit the Newton law $m\mathbf{a} = \mathbf{F} + \mathbf{\Phi}$ holds, we necessarily deduce $\mathbf{\Phi} = -\mathbf{F}$: that is, the floor reacts to the weight, by producing in some way a force (constraint reaction) $\mathbf{\Phi}$ which exactly balances the weight. We thus see that the constraint reaction "adapts" to the actual strenght, and is not known in advance, or deducible in some way by knowing in detail the structure of the floor. Moreover: in general, $\mathbf{\Phi}$ also depends on the velocity of the point $P$.[7] This is evident, for example, in the simple case of a point mass rotating uniformly on a horizontal circle, subject to the gravity $\mathbf{F} = -mg\mathbf{e}_z$: as is known, in such a case the acceleration is purely radial ($a_\vartheta = 0$) and it is $a_r = -r\dot{\vartheta}^2$; so, from $m\mathbf{a} = \mathbf{F} + \mathbf{\Phi}$ it follows $\mathbf{\Phi} = -m\dot{\vartheta}^2 r\mathbf{e}_r - \mathbf{F}$.

An easy balance between the number of equations at our disposal (three, corresponding the Newton vector equation) and the number of unknowns (three for $\mathbf{\Phi}$, $n$ for the motion) shows the problem is undetermined, unless we have at our disposal some further informations about the device that realizes the constraint; for the constraint to a surface or to a curve, the number of lacking equations is, respectively, $n = 2$ and $n = 1$. The Lagrangian method applies to the case of the so-called *ideal constraints,* and allows to write in all cases exactly $n$ equations for the movement $P = P(t)$, in which the constraint reaction $\mathbf{\Phi}$ does not appear explicitly; once the motion $\mathbf{w}(t)$ has been determined, and with it the acceleration $\mathbf{a}(t)$, the constraint reaction, if required, is easily determined by the relation $\mathbf{\Phi} = m\mathbf{a} - \mathbf{F}$.

A general definition of ideal constraint will be provided below in section 2.2.3. In the simple case considered here of a point mass constrained to a given surface or curve, the constraint is said to be ideal if the susface or the curve are smooth, namely the reaction $\mathbf{\Phi}$ in $\mathbf{w} \in \mathcal{Q}$ is always exactly perpendicular to the surface or to the curve in $\mathbf{w}$ (figures 2.3 and 2.4). Recalling the notation $\delta\mathbf{w}$ for a generic tangent vector to $\mathcal{Q}$ in $\mathbf{w}$, the above condition expresses analytically in the form

$$\mathbf{\Phi} \cdot \delta\mathbf{w} = 0 \tag{2.1.11}$$

for any tangent vector $\delta\mathbf{w}$, or equivalently through the $n$ conditions ($n = 1$ or 2)

$$\mathbf{\Phi} \cdot \frac{\partial \mathbf{w}}{\partial q_h} = 0 \qquad \forall h \ . \tag{2.1.12}$$

It appears then clearly that the assumption the constraint be ideal produces exactly as many equations as is necessary to make the problem determined. Moreover, if we proceed as for the free point mass, that is we project the Newton equation along each of the tangent vectors $\frac{\partial \mathbf{w}}{\partial q_h}$, the constraint reaction $\mathbf{\Phi}$ completely disappears, and we are left with exactly $n = 1$ or $n = 2$ equations for the movement, which are

$$(m\mathbf{a} - \mathbf{F}) \cdot \frac{\partial \mathbf{w}}{\partial q_h} = 0 \qquad \forall h \ . \tag{2.1.13}$$

[7] The fact that $\mathbf{\Phi}$ is not purely positional, and depends in general on the motion (typically, as in the following example, on the square of the velocity), raises nontrivial conceptual problems, if one aims to construct a physical model of constraint, that is a device which not by magic, but obeying to the laws of Physics, is able to produce in each circumstance the reaction which is necessary to make the motion to conform to the constraint. For an introduction to this problem (generally neglected in the literature, and not even mentioned in classical textbooks), see V.I. Arnold, *Mathematical Methods of Classical Mechanics,* Springer 1978, and G. Gallavotti, *The elements of Mechanics,* Springer 1983. The fundamental idea however, although difficult to reconcile with the necessity of forces depending on velocities, is the intuitive one: the constraint is realized by some device having high rigidity (large elastic constants, ideally infinite), able to react with arbitrarily large forces to imperceptible deformations.

By comparison with (2.1.5), which formally coincides with (2.1.13) if we let $n = 3$, we see that also for dynamics, as for kinematics, the problem of the point mass constrained to a smooth surface or curve is analog to the problem of the free point mass described in arbitrary coordinates, only the number $n$ of degrees of freedom being different in the different cases.

### 2.1.3 Relative motions and mobile constraints

The above considerations concerning the motion a single point mass, easily extend to two other interesting problems. An especially relevant one is the problem of relative motions: how to write the correct equations of motion in a mobile reference frame, for example a frame which rotates uniformly, with given angular velocity $\omega$, around the $z$ axis of the fixed frame. As we shall see in detail in the next section, in Lagrangian mechanics it is not necessary to introduce the so-called "fictitious forces", like the centrifugal or the Coriolis forces, so as to simulate we are in an inertial reference frame. It will be enough to write a time-dependent change of coordinates, in the example

$$x = q_1 \cos \omega t - q_2 \sin \omega t, \qquad y = q_1 \sin \omega t + q_2 \cos \omega t, \qquad z = q_3 \ ,$$

and proceed exactly as above: that is projecting the Newton equation, point by point, along the three vectors $\frac{\partial \mathbf{w}}{\partial q_h}$ tangent to the coordinate lines (frozen in their instantaneous configuration), regardless of the fact that coordinate lines do move. The correct equations of motion will automatically follow, as in the time independent case.

A second problem, clearly related to the previous one, is that of mobile constraints: for example a spherical pendulum of radius $R$, whose suspension point $C$ is not fixed, but moves along the vertical axis with some given law $z_C = z_C(t)$; the equation of such a constraint has the form $F(x, y, z, t) = 0$, with

$$F(x, y, z, t) = x^2 + y^2 + (z - z_C(t))^2 - R^2 \ ,$$

while the parametric representation of the motion, using as before the angles $\vartheta, \varphi$ of the spherical coordinates, is

$$x = R \sin \vartheta \cos \varphi \ , \qquad y = R \sin \vartheta \sin \varphi \ , \qquad z = z_C(t) + R \cos \vartheta \ .$$

There is no essential difference with respect to the case of fixed constraints: here too, it will be enough to project the Newton equation along the two directions tangent to the sphere in its instanteneous configuration, that is along the two vectors $\frac{\partial \mathbf{w}}{\partial \vartheta}$, $\frac{\partial \mathbf{w}}{\partial \varphi}$ tangent to the coordinate lines, as if the constraint were fixed. The notion of ideal constraint, too, remains unchanged: the constraint reaction $\mathbf{\Phi}$ is assumed to be perpendicular to the sphere, in its instantaneous configuration, ignoring the fact that the sphere moves.

## 2.2 Systems of $N$ possibly constrained point masses

### 2.2.1 Holonomic constraints and free coordinates

After having illustrated in the previous section the aim of the Lagrangian formalism, and having introduced, for the case of a single point mass, the notion of constraint and of free coordinates, we come now to the general case of $N$ point masses ($N$ finite); with minor additional work, we shall include the possibility of time dependent coordinates and of mobile constraints. Consider a system of $N$ point masses $P_1, \ldots, P_N$, and introduce for their Cartesian coordinates the compact notation

$$\mathbf{w} \ = \ (w_1, \ldots, w_{3N}) \ = \ (x_1, y_1, z_1, \ldots, x_N, y_N, z_N) \ ,$$

so as a single vector $\mathbf{w} \in \mathbb{R}^{3N}$ provides a complete description of the whole system.

In absence of constraints, a generic change of coordinates from $\mathbf{w}$ to new coordinates $\mathbf{q} = (q_1, \dots, q_n)$, $n = 3N$, possibly dependent on time, writes

$$\mathbf{w} = \mathbf{w}(\mathbf{q}, t) \ .$$

The jacobian matrix $\frac{\partial w_i}{\partial q_h}$ is supposed to have everywhere, and at any time, nonzero determinant; this ensures that, at least locally, the change of coordinates is a diffeomorphism. The same condition also ensures that the vectors $\frac{\partial \mathbf{w}}{\partial q_h}$, $h = 1, \dots, n$, which are tangent to the $n$ coordinate lines, are linearly independent, and thus provide a local basis in $\mathbb{R}^{3N}$.

Coming now to constrained systems, let as first give a definition:

DEFINITION 10 *We shall say a system of $N$ point masses $P_1, \dots, P_N$ is subject to $r$* holonomic *constraints, $1 \le r < 3N$, if the set of the configurations $\mathbf{w}$ the system can access satisfies $r$ equations of the form*

$$F^{(s)}(\mathbf{w}, t) \ = \ 0 \ , \qquad s = 1, \dots, r \ , \tag{2.2.1}$$

*where $F^{(1)}, \dots, F^{(r)}$ are regular independent functions, such that*

$$\operatorname{rank} \Big( \frac{\partial F^{(s)}}{\partial w_j} \Big) = r \tag{2.2.2}$$

*for any accessible configuration, that is wherever condition (2.2.1) is satisfied.*

In this way, at any time $t$ we deal with a manifold $\mathcal{Q} \subset \mathbb{R}^{3N}$ of dimension $n = 3N - r$ (actually a submanifold of $\mathbb{R}^{3N}$); $\mathcal{Q}$ is called the *constraint manifold,* or *configuration space,* while $n$ is called the *number of degrees of freedom* of the constrained system.

According to mathematical analysis, condition (2.2.1) allows to introduce, at least locally, a parametric representation of $\mathcal{Q}$, that is to express all coordinates $w_1, \dots, w_{3N}$ as functions of $n$ convenient parameters $q_1, \dots, q_n$, called *free coordinates,* and of $t$: precisely,[8]

$$w_i = w_i(q_1, \dots, q_n, t) \ , \qquad i = 1, \dots, 3N \ , \tag{2.2.3}$$

with

$$\operatorname{rank} \Big( \frac{\partial w_i}{\partial q_h} \Big) = n \ . \tag{2.2.4}$$

The parametrization in general is only local, that is covers only a portion of $\mathcal{Q}$, providing, in the common language of geometry, a *chart* of $\mathcal{Q}$. (The terminology clarly comes from geography, where a chart reproduces a portion of the Earth's surface on a portion of plane.) Should it be required, it is always possible to cover $\mathcal{Q}$ with different charts, overlapping on the borders, so constructing an *atlas* of $\mathcal{Q}$ (like the common geographical atlases). Only exceptionally it is possible to have a single global parametric representation of $\mathcal{Q}$, with a single chart. In the following, for simplicity, we shall stick to the local point of view, but it is important to know that everything could be extended globally.

The meaning of property (2.2.4) is that the $n$ vectors

$$\frac{\partial \mathbf{w}}{\partial q_h} \ , \qquad h = 1, \dots, n \ ,$$

[8] Indeed, in virtue of the implicit function theorem, we can always take as free coordinates $n$ conveniently chosen coordinates out of $w_1, \dots, w_{3N}$. A similar choice, however, it is not always the most convenient one.

which are tangent to the coordinate lines in a generic point $\mathbf{w} \in \mathcal{Q}$, provide a basis for the tangent space to $\mathcal{Q}$ in $\mathbf{w}$. A generic vector tangent to $\mathcal{Q}$, in its instantaneous configuration at any time $t$, will be denoted $\delta\mathbf{w}$, and we shall write

$$\delta\mathbf{w} = \sum_{h=1}^{n} \frac{\partial \mathbf{w}}{\partial q_h} \delta q_h \ , \tag{2.2.5}$$

where $\delta q_h$, $h = 1, \ldots, n$, are arbitrary coefficients.

Quite in general, keeping together the constrained and the unconstrained cases, it is convenient to make reference to the following

DEFINITION 11 *A system of $N$ point masses is said to be a* local holonomic system *with $n \leq 3N$ degrees of freedom, if its configuration $\mathbf{w} \in \mathbb{R}^{3N}$ is expressed locally in the parametric form (2.2.3), by means of regular functions $w_1, \ldots, w_{3N}$ satisfying (2.2.4).*

The case $n = 3N$ evidently corresponds to a change of coordinates, with no constraint ($r = 0$); in such a case the constraint manifold $\mathcal{Q}$ is the whole space, and the tangent space too coincides with the whole $\mathbb{R}^{3N}$. In the unconstrained case, the expression (2.2.5) simply expresses a generic vector of $\mathbb{R}^{3N}$ in the local basis $\frac{\partial \mathbf{w}}{\partial q_h}$, $h = 1, \ldots, n$, formed by the vectors tangent to the $n = 3N$ coordinate lines.

Making reference, as will also be useful, to the common notation $P_1, \ldots, P_N$ for the $N$ point masses, the parametric representation assumes the form, equivalent to (2.2.3),

$$P_i \ = \ P_i(q_1, \ldots, q_n, t) \ , \qquad i = 1, \ldots, N \tag{2.2.6}$$

(this is nothing but taking three by three the components of $\mathbf{w}$). Correspondingly, we can introduce the *virtual displacement of the $i$-th point*

$$\delta P_i = \sum_{h=1}^{n} \frac{\partial P_i}{\partial q_h} \delta q_h \ , \qquad i = 1, \ldots, N \ , \tag{2.2.7}$$

which is nothing but a triplet of components of $\delta\mathbf{w}$ (by aligning one after the other the $N$ virtual displacement of the $N$ points, the tangent vector $\delta w$ is reconstructed: that is, $\delta\mathbf{w} = (\delta w_1, \ldots, \delta w_{3N}) = (\delta P_1, \ldots, \delta P_N)$).

EXAMPLE Consider a system of two points $P_1$, $P_2$, constrained to stay at a fixed distance $d$; the constraint can be expressed in the form $F = 0$, with $F = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - d^2$. There are then $n = 5$ free coordinates. A possible choice is provided by the three Cartesian coordinates of $P_1$ and the two angles $\vartheta$ and $\varphi$ of the spherical coordinates of the vector $(P_2 - P_1)$. A more convenient choice, in view of the equations of motion that follow, is taking, in place of the Cartesian coordinates of $P_1$, the Cartesian coordinates of the barycenter.

### 2.2.2 Non-holonomic constraints (a short comment)

For the sake of completeness, we just mention here the so-called *non-holonomic* constraints, if only for explaining the above introduced terminology. Let us consider the case of a single constraint, of equation $F(\mathbf{w}) = 0$ (for simplicity, we assumed the constraint to be fixed). By differentiating, we obtain the relation

$$f_1(\mathbf{w})\, \mathrm{d}w_1 + \cdots + f_{3N}(\mathbf{w})\, \mathrm{d}w_{3N} = 0 \ , \tag{2.2.8}$$

with $f_i = \frac{\partial F}{\partial w_i}$; in such a form, the constraint appears as a limitation to the allowed displacements, rather than to the possible configurations. Such an equation including differentials is however substantially equivalent to the finite or integer relation $F = 0$. (More precisely, the differential form (2.2.8) corresponds to a family of constraints of the form $F(\mathbf{w}) = 0$, the different $F$ differing by an additive constant.)

Quite obviously, not any limitation to the displacements of the form (2.2.8) gives, by integration, an equation in integer form: for this to happen (at least locally), the closure relations

$$\frac{\partial f_i}{\partial w_j} = \frac{\partial f_j}{\partial w_i} \ , \qquad i, j = 1, \dots, 3N,$$

need to be satisfied, up to a possible previous multiplication of the $f_i$'s by a convenient common integrating factor[9] $\lambda(\mathbf{w})$. In such a case the constraint is still called holonomic (or also *semi-holonomic*). In fact, the word "holonomic" means precisely the constraint can be expressed by an "integer form" (from the greek *ólos* = whole or integer, *nómos* = law).

The differential expression (2.2.8) can be equivalently rewritten by using velocities, precisely

$$f_1 \dot{w}_1 + \cdots + f_{3N} \dot{w}_{3N} \ ;$$

the constraint then appears as a limitation to the admitted velocities.

An example of constraint that is spontaneously written in the differential form, but turns out to be holonomic, is the condition of pure rolling of a disc on a line. Consider a disc of radius $R$ leaning on the $x$ axis; its configuration is determined by the two coordinates $x$ and $\varphi$, where $x$ is the abscissa of the center $C$ of the disc, actually the same as that of the contact point $A$ of the disc with the $x$ axis, while $\varphi$ is the angle from the segment $OC$ and some fixed radius drawn in the disc. If however we impose the constraint of pure rolling (that is, point $A$ of the disc has zero velocity), there is the further condition

$$\mathrm{d}x + R\mathrm{d}\varphi = 0 \ , \qquad \text{equivalently} \qquad \dot{x} + R\dot{\varphi} = 0 \ .$$

The differential form representing the constraint is however closed, and the constraint can be written in the integer form

$$x + R\varphi - c = 0 \ ,$$

$c$ being an arbitrary constant (determined, observe, once an initial datum is given). The system is then a holonomic system with one degree od freedom, and either $x$ or $\varphi$ can be conveniently taken as the free coordinate.

A few simple examples of non-holonomic constraints are sketched in appendix A.

### 2.2.3 Ideal constraints

*A. The notion of ideal constraint.* Let us consider a holonomic system with $n$ degrees of freedom composed by $N$ point masses $P_1, \dots, P_N$, and let (2.2.3) be the parametric equations providing the position of each point as a function of the free coordinates $q_1, \dots, q_n$ and of time. Determining the movement $P_i(t)$, $i = 1, \dots, N$, or equivalently $\mathbf{w}(t)$, means determining the $n$ unknown functions $q_h(t)$, $h = 1, \dots, n$. We assume the system moves respecting the constraints, being subject to some

[9] Indeed, (2.2.8) is equivalent to $g_1(\mathbf{w})\mathrm{d}w_1 + \cdots + g_{3N}(\mathbf{w})\mathrm{d}w_{3N} = 0$, where $g_j(\mathbf{w}) = \lambda(\mathbf{w}) f_j(\mathbf{w})$, $i = 0, \dots, 3N$, $\lambda(\mathbf{w}) \neq 0$.

given *active forces* $\mathbf{F}_i$, one for each point, and to suitable *constraint reactions* $\mathbf{\Phi}_i$, also one for each point. Active forces are supposed to be known functions of positions and velocities of all points, and possibly of time:

$$\mathbf{F}_i(P_1, \ldots, P_N, \mathbf{v}_1, \ldots, \mathbf{v}_N, t) \qquad \text{or} \qquad \mathbf{F}_i(\mathbf{w}, \dot{\mathbf{w}}, t) \ .$$

The constraint reactions instead are not known in advance: we only know they do whatever is needed, for the motion to obey the constraint. The Newton equations for the $N$ point masses write then:

$$m_i \mathbf{a}_i \ = \ \mathbf{F}_i + \mathbf{\Phi}_i \ , \qquad i = 1, \ldots, N \ , \tag{2.2.9}$$

and we are confronted with the problem of deducing from them a system of $n$ "pure" equations, namely equations that do not contain the constraint reactions, for the unknowns $q_1, \ldots, q_n$, so as the motion is determined.

A trivial count of the number of equations and unknowns shows the problem is undetermined, unless we impose some restriction on the constraint reactions $\mathbf{\Phi}_i$. Indeed, the (2.2.9) are a system of only $3N$ scalar equations, while the unknowns, also including the $3N$ components of the constraint reactions, are $3N + n$. The $n$ lacking equations cannot be searched in the principles of mechanics, that we already used, and must come from an *a priori* assumption on the kind of forces the physical device providing the constraint reactions is able to produce. This should not surprise us, if we think that a point mass constrained, for example, to a surface, certainly moves differently (although respecting the constraint, in presence of the same active forces) if the surface is smooth or differently rough.

Let us then come to the notion of *ideal constraint,* which generalizes to the case of several point masses the idea we already introduced for the case of a single point mass.

Definition 12 *We shall say a holonomic system of $N$ point masses is subject to ideal constraints, if the set of the a priori admissible constraint reactions $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_N$ is characterized by the condition*

$$\sum_{i=1}^{N} \mathbf{\Phi}_i \cdot \delta P_i \ = \ 0 \tag{2.2.10}$$

*for any choice of the virtual displacements $\delta P_1, \ldots \delta P_N$.*

The left hand side of (2.2.10) is traditionally called the *virtual work* of the constraint reactions; the condition the constraint is ideal can then be stated by saying the device that physically realizes the constraint is able to produce all and only forces $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_N$ that perform null virtual work, for all conceivable virtual displacements.

Using (2.2.7), also taking into account the arbitrariness of the $\delta q_h$, equation (2.2.10) results in the independent conditions

$$\sum_{i=1}^{N} \mathbf{\Phi}_i \cdot \frac{\partial P_i}{\partial q_h} = 0 \ , \qquad h = 1, \ldots, n \ , \tag{2.2.11}$$

which are exactly $n$, that is as many as required to make the problem determined.[10]

[10] Observe that, in absence of constraints, the virtual displacements $\delta P_1, \ldots, \delta P_N$ are themselves independent; from (2.2.11) we deduce then, as expected, that all constraint reactions vanish.

- Equation (2.2.10) has a strong analogy with the case of a single point mass. The analogy becomes clear we introduce the vector

$$\mathbf{\Psi} = (\Psi_1, \dots, \Psi_{3N}) = (\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_N) \in \mathbb{R}^{3N} \ ,$$

in which the components of the constraint reaction are placed one after the other, exactly as the coordinates of the points do compose $\mathbf{w}$. Using this notation, the ideality condition writes

$$\mathbf{\Psi} \cdot \delta\mathbf{w} = 0 \qquad \text{for any choice of the tangent vector } \delta\mathbf{w} \ ,$$

or equivalently

$$\mathbf{\Psi} \cdot \frac{\partial \mathbf{w}}{\partial q_h} = 0 \ , \qquad h = 1, \dots, n \ .$$

We then see that *in the case of $N$ point masses, too, the ideality of the constraint can be interpreted as the orthogonality of the constraint reaction, more precisely of $\mathbf{\Psi}$, to the constraint manifold $\mathcal{Q}$.*

In a completely equivalent way, in the notion of ideal constraint we can make reference to all possible velocities $\mathbf{u}_i$ of the point masses, which are *a priori* compatible with the constraints: namely

$$\mathbf{u}_i = \sum_{h=1}^{n} \frac{\partial P_i}{\partial q_h} \dot{q}_h \ ,$$

with arbitrary $\dot{q}_1, \dots \dot{q}_n$; the $\mathbf{u}_i$ are called *virtual velocities.*[11] In this language, we can say the constraints are ideal if the power of the constraint reactions vanishes for all choices of the virtual velocities $\mathbf{u}_1, \dots, \mathbf{u}_N$:

$$\sum_{i=1}^{N} \mathbf{\Phi}_i \cdot \mathbf{u}_i = 0 \ , \qquad \forall \ \mathbf{u}_1, \dots, \mathbf{u}_N \ . \tag{2.2.12}$$

- Concerning the definition of ideal constraint, it is worthwhile to stress a fact that, at first glance, might escape. Consider for simplicity the case of a single point mass constrained to a fixed surface. The ideality condition implies that, as we just observed, for any effective motion, the constraint reaction does not do work. However, the assumption that the work vanishes for any effective motion, does not imply that the constraint is ideal. Indeed, for the effective work to vanish it is enough the constraint reaction aroused by each particular motion (which depends on the motion) is orthogonal to that motion only. The ideality condition requires instead that the constraint reaction is orthogonal to the surface, that is orthogonal not only to the velocity of the motion at hand, but simultaneously to all velocities tangent to the surface (think of a magnetic force: no work, but not orthogonal to the surface).

*B. The ideality of the rigidity constraint.* An important example of ideal constraint is the rigidity constraint, namely the requirement that for any pair of points $P_i$, $P_j$ of the body the distance $\|P_i - P_j\|$ stays constant. (For basic notions about Newtonian mechanics of rigid systems, here

[11] In the case of fixed constraints, the effective velocities $\mathbf{v}_i$, realized in a certain motion, are indeed a particular choice inside the set of virtual velocities $\mathbf{u}_i$. In the case of mobile constraints, this is not anymore true, for the presence, in the effective motion, of non tangent components of the velocity (think, for example, of a point at rest on a plane that rises).

assumed to be familiar, see appendix B). More precisely, it is not difficult to prove that the rigidity constraint is ideal, if we assume it be physically realized by pairs of internal forces satisfying Newton's third law (pairs of opposite central forces):

PROPOSITION 17 *Consider a rigid system of $N$ point masses $P_1, \dots, P_N$, and let $\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_N$ be the constraint reactions associated to a given motion. Assume*

$$\mathbf{\Phi}_i = \sum_{j \neq i} \mathbf{F}_{i,j} \ ,$$

*the forces $\mathbf{F}_{i,j}$ satisfying Newton's third law:*

$$\mathbf{F}_{i,j} = -\mathbf{F}_{j,i} \ , \qquad \mathbf{F}_{i,j} \quad \text{parallel to} \quad (P_i - P_j) \ .$$

*Then the constraint is ideal.*

PROOF. We limit ourselves, for simplicity, to the case of two point masses, the generalization being trivial. The assumption on the constraint reactions reduces, in such a case, to

$$\mathbf{\Phi}_2 = -\mathbf{\Phi}_1 \ , \qquad \mathbf{\Phi}_1 \quad \text{parallel to} \quad (P_1 - P_2) \ .$$

By definition of rigid system, it is $\frac{\mathrm{d}}{\mathrm{d}t}(P_1 - P_2)^2 = 0$, so for any possible motion, that is for any choice of velocities $\mathbf{u}_1, \mathbf{u}_2$ compatible with the constraint, it is $(P_1 - P_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) = 0$; it immediately follows that the power of the constraint reactions vanishes:

$$\mathbf{\Phi}_1 \cdot \mathbf{u}_1 + \mathbf{\Phi}_2 \cdot \mathbf{u}_2 = \mathbf{\Phi}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) = 0 \ .$$

□

It is important to observe that *the assumptions on the forces that realize the constraint are the same ones that guarantee the validity of the cardinal equations*

$$\dot{\mathbf{Q}} = \mathbf{R}^{\mathrm{ext}} \ , \qquad \dot{\mathbf{M}} = \mathbf{N}^{\mathrm{ext}} \ ,$$

where $\mathbf{Q}$ and $\mathbf{M}$ denote, respectively, the momentum and the angular momentum of the system, while $\mathbf{R}^{\mathrm{ext}}$ denotes the resultant of the external forces (of the active forces, in the Lagrangian language) and $\mathbf{N}^{\mathrm{ext}}$ is their resultant torque. In fact, it is possible to prove (see appendix B) that *for rigid bodies, cardinal equations are equivalent to the ideality of the constraint.* We shall come back on this point after introducing the Lagrange equations. The fact that the rigidity constraint is ideal, that is the power of the constraint reactions vanishes, corresponds to the fact, well known within Newtonian Mechanics, that for rigid bodies (at variance with generic non–rigid systems of $N$ point masses) internal forces do not contribute to the potential energy.

- The example of the rigidity constraint shows in particular how important is asking, in (2.2.10) or in (2.2.12), that the overall virtual work (or power) vanishes, without assuming $\mathbf{\Phi}_i \cdot \delta P_i = 0$ (or $\mathbf{\Phi}_i \cdot \mathbf{u}_i = 0$) separately for each point.

## 2.3 Kinetic energy, work and potential energy

Our purpose here is to determine the expression of kinetic energy, of work and (for conservative systems) of potential energy, as functions of the free coordinates $\mathbf{q} = (q_1, \dots, q_n)$ and of the corresponding velocities $\dot{\mathbf{q}} = (\dot{q}_1, \dots, \dot{q}_n)$, also called *generalized velocities.*

### 2.3.1 Kinetic energy

The parametric equations (2.2.3) and (2.2.6) provide the configuration $\mathbf{w}$ of the system and the position of each point $P_i$ as functions of $\mathbf{q}$ and possibly of time; correspondingly, for the velocities $\dot{w}_i$, $i = 1, \dots, 3N$ and $\mathbf{v}_i$, $i = 1, \dots, N$, we have

$$\dot{w}_i = \sum_{h=1}^{n} \frac{\partial w_i}{\partial q_h}(\mathbf{q}, t)\, \dot{q}_h + \frac{\partial w_i}{\partial t}(\mathbf{q}, t)\ , \qquad \mathbf{v}_i(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{h=1}^{n} \frac{\partial P_i}{\partial q_h}(\mathbf{q}, t)\, \dot{q}_h + \frac{\partial P_i}{\partial t}(\mathbf{q}, t) \tag{2.3.1}$$

(observe that in the time dependent case, the vector $\dot{\mathbf{w}}$ is *not* tangent to the constraint manifold, because of the term $\frac{\partial \mathbf{w}}{\partial t}$).

Concerning the kinetic energy of the system, the following easy proposition holds:

PROPOSITION 18 *For a holonomic system of $N$ point masses, the kinetic energy, expressed as a function of the free coordinates, has the form*

$$K = K_2 + K_1 + K_0\ , \tag{2.3.2}$$

*$K_2$, $K_1$ and $K_0$ being homogeneous polynomial of degree, respectively, 2, 1 and 0 in $\dot{\mathbf{q}}$:*

$$K_2 = \frac{1}{2} \sum_{h,k=1}^{n} a_{hk}(\mathbf{q}, t)\, \dot{q}_h \dot{q}_k\ , \qquad K_1 = \sum_{h=1}^{n} b_h(\mathbf{q}, t)\, \dot{q}_h\ , \qquad K_0 = \frac{1}{2}\, c(\mathbf{q}, t)\ . \tag{2.3.3}$$

*The coefficients $a_{hk}$, $b_h$, $c$ are given by*

$$a_{hk} = \sum_{i=1}^{N} m_i \frac{\partial P_i}{\partial q_h} \cdot \frac{\partial P_i}{\partial q_k}\ , \qquad b_h = \sum_{i=1}^{N} m_i \frac{\partial P_i}{\partial q_h} \cdot \frac{\partial P_i}{\partial t}\ , \qquad c = \sum_{i=1}^{N} m_i \frac{\partial P_i}{\partial t} \cdot \frac{\partial P_i}{\partial t}\ . \tag{2.3.4}$$

*Finally, matrix $a = (a_{hk})$ is symmetric and positive definite.*[12]

Matrix $a$ is called the *kinetic matrix.* The fact $a$ is positive definite plays a quite relevant role throughout Lagrangian mechanics. A particularly important consequence, that we shall invoke several times, is that the determinant of $a$ is different from zero (actually is positive, see footnote 12), and thus $a$ is invertible. In the particular but important case in which equations (2.2.3), (2.2.6) do not explicitly contain time (time independent change of coordinates, fixed constraints), it is $K_1 = K_0 = 0$, so that $K$ is a (positive definite) homogeneous quadratic form.

PROOF. By replacing expression (2.3.1) of $\mathbf{v}_i$ in the usual formula $\frac{1}{2}\sum_{i=1}^{N} m_i v_i^2$ of the kinetic energy, we find

$$K = \frac{1}{2} \sum_{i=1}^{N} m_i \Big[ \sum_{h,k=1}^{n} \frac{\partial P_i}{\partial q_h} \cdot \frac{\partial P_i}{\partial q_k}\, \dot{q}_h \dot{q}_k + 2 \sum_{h=1}^{n} \frac{\partial P_i}{\partial q_h} \cdot \frac{\partial P_i}{\partial t}\, \dot{q}_h + \frac{\partial P_i}{\partial t} \cdot \frac{\partial P_i}{\partial t} \Big]\ ;$$

[12] Let us recall that a symmetric matrix $A$ is said to be positive definite, if for any $\mathbf{u} \neq 0$ it is $A\mathbf{u} \cdot \mathbf{u} > 0$; it is possible to prove that $A$ is positive definite if and only if all diagonal minors (including $A$ itself) have positive determinant. From this property, but also directly from the definition, it follows that if $A$ is positive definite, then any diagonal minor is also positive definite. Another necessary and sufficient condition is that all eigenvalues of $A$ (which for symmetric $A$ are real) are positive. For a symmetric $2 \times 2$ matrix $A$ it is immediate to see that $A$ is positive definite if $\det A > 0$ and the diagonal entries (which for $\det A > 0$ have the same sign) are positive. Instead, $\det A > 0$ is not enough to ensure that $A$ is positive definite, as is shown by the trivial counterexample

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

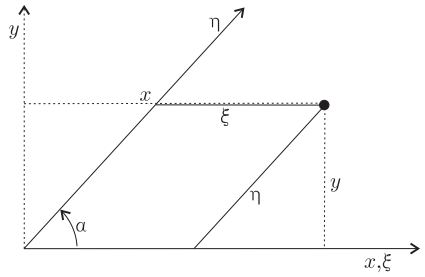


Figure 2.5: *The oblique Cartesian velocities* $\xi, \eta$.

exchanging the sums and using (2.3.4), the claimed expression of $K$ immediately follows.

Concerning the properties of the kinetic matrix $a$, its symmetry trivially follows from the symmetry of the scalar product. To show $a$ is positive definite, we must show that $K_2 > 0$ for any choice of $\dot{\mathbf{q}} \neq 0$. For this purpose, let us introduce the velocities "at frozen time"

$$\mathbf{v}_i^* = \sum_{h=1}^{n} \frac{\partial P_i}{\partial q_h}\, \dot{q}_h \ , \qquad v_i = v_i^* + \frac{\partial P_i}{\partial t} \ .$$

As is evident, $K_2 = \frac{1}{2}\sum_{i=1}^{N} m_i (\mathbf{v}_i^*)^2$. But in virtue of condition (2.2.4) on the rank of the matrix $\frac{\partial w_i}{\partial q_h}$, we know that if $\dot{\mathbf{q}} \neq 0$, then the velocities $\mathbf{v}_i^*$ cannot be all equal to zero; as a consequence, $K_2 > 0$. □

Exercise 26 *Write the kinetic energy of a point mass, in plane and spherical polar coordinates.* Answere: for plane polar coordinates it is

$$K(r, \vartheta, \dot{r}, \dot{\vartheta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2) \ ; \tag{2.3.5}$$

for spherical coordinates it is

$$K(\rho, \vartheta, \varphi, \dot{\rho}, \dot{\vartheta}, \dot{\varphi}) = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2 + \rho^2 \sin^2\vartheta\dot{\varphi}^2) \ . \tag{2.3.6}$$

Exercise 27 *Write the kinetic energy of the spherical pendulum of length* $R$. Answer:

$$K = \frac{1}{2}mR^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)$$

(it is enough to consider the expression of $K$ in spherical coordinates, and impose the further constraint $\rho = R$).

In the above exercises, the kinetic energy does not contain mixed times in the velocities (the kinetic martix is diagonal). It is easy to see this is due to the fact the coordinate lines intersect othogonally. Instead:

Exercise 28 *Consider a point mass costrained to a plane; in place of the usual* $x$ *and* $y$ *axes, use an axis* $\xi$ *coinciding with* $x$ *and an axis* $\eta$ *forming a given angle* $\alpha$ *with the* $\xi$ *axis (figure 2.5). Write the kinetic energy.* Answere: from the easy relations $x = \xi + \eta\cos\alpha$, $y = \eta\sin\alpha$, it immediately follows

$$K = \frac{1}{2}m(\dot{\xi}^2 + \dot{\eta}^2 + 2\cos\alpha\,\dot{\xi}\dot{\eta}) \ . \tag{2.3.7}$$

EXERCISE 29 *Write the kinetic energy of a system of two point masses $P_1$, $P_2$, constrained to stay on a plane and to mantain fixed distance $d$; use as free coordinates: (a) the Cartesian coordinates $x_1, y_1$ of $P_1$ and the angle $\vartheta$ between the $x$ axis and the vector $P_2 - P_1$; (b) the coordinates $X, Y$ of the barycenter and the same angle $\vartheta$.* Answere: in case (a) we find

$$K = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2[d^2\dot{\vartheta}^2 - 2d(\dot{x}_1 \sin\vartheta - \dot{y}_1 \cos\vartheta)\dot{\vartheta}] \ ,$$

with $m = m_1 + m_2$; in case (b) we find the much simpler expression

$$K = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu\, d^2\dot{\vartheta}^2 \ ,$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$; $\mu$ is called the *reduced mass* of the system. It is like having a mass $m$ free to move on the plane and a second mass $\mu$ constrained to a circle of radius $d$, independent from each other.

EXERCISE 30 *Write the kinetic energy of a point mass constrained to the $xy$ plane, in a Cartesian frame $XY$ that rotates uniformly, with angular velocity $\omega$, around the $z$ axis: that is,*

$$x(X, Y, t) = X\cos\omega t - Y\sin\omega t \ , \qquad y(X, Y, t) = X\sin\omega t + Y\cos\omega t \ .$$

Answer: the expression is $K = K_2 + K_1 + K_0$, with

$$K_2 = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) \ , \qquad K_1 = m\omega(X\dot{Y} - \dot{X}Y) \ , \qquad K_0 = \frac{1}{2}m\omega^2(X^2 + Y^2) \ .$$

EXERCISE 31 *Write the kinetic energy of the* centrifugal pendulum, *that is a pendulum whose plane of oscillation rotates uniformly, with angular velocity $\omega$, around the vertical axis passing through the suspension point.* Answer: it is enough to consider the kinetic energy of the spherical pendulum, and impose the further constraint $\varphi = \omega t$ (and consequently $\dot{\varphi} = \omega$); the result is

$$K = \frac{1}{2}mR^2(\dot{\vartheta}^2 + \omega^2\sin^2\vartheta) \ .$$

Other exercises focused on the calculation of kinetic energy can be found in the exercise part of these lecture notes.

### 2.3.2 Forces, work, potential energy

Consider a holonomic system, and let $\mathbf{F}_i$ be the active force on the $i$-th point mass, $i = 1, \ldots, N$; the quantity

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta P_i \ ,$$

that is the work of the active forces corresponding to the virtual displacements $\delta P_1, \ldots, \delta P_N$, is called the *virtual work* of the active forces.

PROPOSITION 19 *It is*

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta P_i = \sum_{h=1}^{n} Q_h \delta q_h \ ,$$

*where*

$$Q_h = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial P_i}{\partial q_h} \ , \qquad h = 1, \ldots, n \ . \tag{2.3.8}$$

PROOF. It is enough to use the (2.2.7) and to exchange the sums. □

In particular, in the case of conservative positional forces, we know there exists a *potential energy function* $V(P_1, \dots, P_N, t)$, such that

$$\mathbf{F}_i = -\nabla_i V \ ,$$

where $\nabla_i$ denotes the gradient restricted to the three coordinates $x_i, y_i, z_i$ of point $P_i$.

PROPOSITION 20 *Denote* $\widehat{V}(\mathbf{q}, t) = V(\mathbf{w}(\mathbf{q}, t), t)$. *It is:*

$$Q_h(\mathbf{q}, t) \ = \ -\frac{\partial \widehat{V}}{\partial q_h} \ , \qquad h = 1, \dots, n \ .$$

PROOF. It is enough to use the standard rule of derivation of composed functions. □

For a single point mass, $Q_h$ is precisely the projection of $\mathbf{F}$ in the direction of the $h$-th coordinate line. But if appropriately understood, this is also true in the case of $N$ point masses: indeed, if we align one after the other all active forces in a single vector

$$\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}_1, \dots, \boldsymbol{\mathcal{F}}_{3N}) = (\mathbf{F}_1, \dots, \mathbf{F}_N) \in \mathbb{R}^{3N} \ ,$$

it is clear that

$$Q_h = \boldsymbol{\mathcal{F}} \cdot \frac{\partial \mathbf{w}}{\partial q_h} \ ;$$

in a similar way, the work is expressed as well by $\boldsymbol{\mathcal{F}} \cdot \delta\mathbf{w}$ or by $\mathbf{Q} \cdot \delta\mathbf{q}$, having denoted $\mathbf{Q} = (Q_1, \dots, Q_n)$, while the relation between forces and potential energy writes

$$\boldsymbol{\mathcal{F}} = -\nabla V = -\Big(\frac{\partial V}{\partial w_1}, \dots, \frac{\partial V}{\partial w_{3N}}\Big) \ ,$$

and correspondingly

$$\mathbf{Q} = -\nabla \widehat{V} = -\Big(\frac{\partial \widehat{V}}{\partial q_1}, \dots, \frac{\partial \widehat{V}}{\partial q_n}\Big) \ .$$

The quantities $Q_1, \dots, Q_n$, which to all purposes replace forces in the Lagrangian formalism, are named *generalized forces,* or *Lagrangian components of the force.*

Without risk of confusion, to simplify the notation we shall denote by $V$, rather than by $\widehat{V}$, the potential energy expressed as a function of the free coordinates.

## 2.4 The Lagrange equations

### 2.4.1 Deducing the equations

Let us consider a holonomic system consisting of $N$ point masses $P_1, \dots, P_N$, of mass $m_1, \dots, m_N$, possibly subject to ideal constraints, moving in some given system of active forces $\mathbf{F}_1, \dots, \mathbf{F}_N$. The Newton equations

$$m_i \mathbf{a}_i \ = \ \mathbf{F}_i + \boldsymbol{\Phi}_i \ , \qquad i = 1, \dots, N \ , \tag{2.4.1}$$

then hold, where $\boldsymbol{\Phi}_i$, $i = 1, \dots, N$, are the constraint reactions; as already discussed, these should be thought of as unknowns. Exploiting the assumption of ideality of the constraints, it is easy to

eliminate the constraint reactions, so as to obtain exactly $n$ pure equations: indeed, multiplying (2.4.1) by $\frac{\partial P_i}{\partial q_h}$ and summing on $i$, we get

$$\sum_{i=1}^{N}(m_i\mathbf{a}_i - \mathbf{F}_i - \mathbf{\Phi}_i)\cdot\frac{\partial P_i}{\partial q_h} = 0\ , \qquad h = 1,\dots,n\ ;$$

thanks to the ideality of the constraint, that is to (2.2.11), the constraint reactions disappear. Using then definition (2.3.8) of $Q_h$, we immediately find

$$\sum_{i=1}^{N} m_i\mathbf{a}_i\cdot\frac{\partial P_i}{\partial q_h} = Q_h\ , \qquad h = 1,\dots,n\ .$$

A fundamental proposition follows:

Proposition 21 *Consider a holonomic system of $N$ point masses, with $n$ degrees of freedom, possibly subject to ideal constraints, which move in an assigned system of active forces $\mathbf{F}_i$, $i = 1,\dots,N$. Then the free coordinates $q_1,\dots,q_n$ satisfy the equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial\dot q_h} - \frac{\partial K}{\partial q_h} = Q_h\ , \qquad h = 1,\dots,n\ . \tag{2.4.2}$$

Equations (2.4.2) will be called the Lagrange equations "in improper form".[13]

Proof. The proof reduces to verify the identity

$$\sum_{i=1}^{N} m_i\mathbf{a}_i\cdot\frac{\partial P_i}{\partial q_h} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial\dot q_h} - \frac{\partial K}{\partial q_h}\ , \qquad h = 1,\dots,n\ , \tag{2.4.3}$$

where $K(\mathbf{q},\dot{\mathbf{q}},t)$, we recall, is the kinetic energy expressed as a function of the free coordinates and of the corresponding generalized velocities, according to (2.3.2)–(2.3.4). Moreover, thanks to the additivity of kinetic energy, that is $K = \sum_{i=1}^{N} K_i$ where $K_i(\mathbf{q},\dot{\mathbf{q}},t)$ is the kinetic energy of the $i$-th point mass (expressed as a function of the free coordinates), it is enough to verify the identity (2.4.3) separately for each $P_i$:

$$m_i\mathbf{a}_i\cdot\frac{\partial P_i}{\partial q_h} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K_i}{\partial\dot q_h} - \frac{\partial K_i}{\partial q_h}\ , \qquad h = 1,\dots,n\ .$$

To this end, from $\mathbf{a}_i = \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t}$ we deduce the identity

$$m_i\mathbf{a}_i\cdot\frac{\partial P_i}{\partial q_h} = m_i\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{v}_i\cdot\frac{\partial P_i}{\partial q_h}\Big) - m_i\mathbf{v}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial P_i}{\partial q_h}\ ; \tag{2.4.4}$$

we can then use the following relations, to be verified below:

$$\frac{\partial P_i}{\partial q_h} = \frac{\partial\mathbf{v}_i}{\partial\dot q_h}\ , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial P_i}{\partial q_h} = \frac{\partial\mathbf{v}_i}{\partial q_h} \tag{2.4.5}$$

[13] Such a terminology is not of common use. We use it here just to distinguish (2.4.2) from the "proper form" Lagrange equations (2.4.7), that we shall soon meet. In the literature, both (2.4.2) and (2.4.7) are commonly called the Lagrange equations, without distinction.

(to remember them: the former is *as if* we divide numerator and denominator by $\mathrm{d}t$; the latter is an exchange of partial and total derivatives). The conclusion is immediate: by replacing (2.4.4) in (2.4.4), it easily follows

$$\begin{aligned} m_i \mathbf{a}_i \cdot \frac{\partial P_i}{\partial q_h} &= m_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_h}\Big) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_h} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_h}\Big(\frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i\Big) - \frac{\partial}{\partial q_h}\Big(\frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K_i}{\partial \dot{q}_h} - \frac{\partial K_i}{\partial q_h} \ . \end{aligned}$$

We now check identities (2.4.5). The former immediately follows from (2.3.1), taking a derivative with respect to $\dot{q}_h$; the latter comes from a quite elementary lemma, that we enucleate here in a general form for a future use:

Lemma 22 *For a generic function $f(\mathbf{q}, t)$, the following identity holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_h} = \frac{\partial}{\partial q_h}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{2.4.6}$$

*(that is, total and partial derivative can be exchanged).*

The easy proof is below. We use here the lemma by taking as $f$, one by one, the coordinates of $P_i$. This concludes the proof of the proposition. □

Proof of the Lemma. Use is made of the formula for the derivation of a composite function:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_h} = \sum_{k=1}^{n} \frac{\partial}{\partial q_k}\frac{\partial f}{\partial q_h}\dot{q}_k + \frac{\partial}{\partial t}\frac{\partial f}{\partial q_h} = \frac{\partial}{\partial q_h}\Big(\sum_{k=1}^{n} \frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial t}\Big) = \frac{\partial}{\partial q_h}\frac{\mathrm{d}f}{\mathrm{d}t}$$

(in essence: the possibility of exchanging the order of partial derivatives, also implies the possibility of exchanging partial and total derivative). □

In the important case of positional forces which admit a potential energy, we deduce immediately, as a corollary, the following

Proposition 23 *For a holonomic system with $n$ degrees of freedom of kinetic energy $K(\mathbf{q}, \dot{\mathbf{q}}, t)$, subject to conservative forces deduced from the potential energy $V = V(\mathbf{q}, t)$, the free coordinates $q_1, \ldots, q_n$ satisfy the equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = 0 \ , \qquad h = 1, \ldots, n \ , \tag{2.4.7}$$

*where $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is defined as*

$$L = K - V \ .$$

Such equations will be called the Lagrange equations "in proper form", or simply the Lagrange equations.

Proof. In these assumptions, as already commented, we have $Q_h = -\frac{\partial V}{\partial q_h}$; since $\frac{\partial V}{\partial \dot{q}_h} = 0$, the conclusion is immediate. □

We see that, in such a fundamental case, the Lagrange equations are deduced from a single function $L$ of coordinates, velocities and time. $L$ is called the *Lagrange function,* or *Lagrangian,* of the system. We shall also use, for the Lagrange equations, the compact notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0 \ .$$

It is also interesting to consider the "mixed" case $Q_h = Q'_h + Q''_h$, where the $Q'_h$ are conservative, namely $Q'_h = -\frac{\partial V}{\partial q_h}$, while the $Q''_h$ are generic. In such a case the Lagrange equations assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = Q''_h \ , \qquad h = 1, \ldots, n \ , \tag{2.4.8}$$

with $L = K - V$.

The quantities $\frac{\partial L}{\partial \dot{q}_h}$ that enter the Lagrange equations, play an important role throughout Lagrangian mechanics. A typical notation, that we shall use in several occasions, is

$$p_h(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{\partial L}{\partial \dot{q}_h}(\mathbf{q}, \dot{\mathbf{q}}, t) \ ; \tag{2.4.9}$$

$p_h$ is called the *momentum* conjugate to $q_h$. The notion of conjugate momentum generalizes the common notion of linear o angular momentum, to which it reduces in the most simple cases.

### 2.4.2 Simple examples

Let us now see some elementary examples of Lagrangian systems.

Example Point mass in Cartesian coordinates $x, y, z$.
The kinetic energy is evidently $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$; if $V(x, y, z)$ is the potential energy, the Lagrangian is

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \ ,$$

and the common Newton equations follow. With evidence, the conjugate momenta $p_x = m\dot{x}\,, \ldots,\ p_z = m\dot{z}$ are the three components of the linear momentum.

Example Point mass in cylindrical coordinates $r, \varphi, z$.
The kinetic energy is $K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2)$. Denoting by $V$ the potential energy, expressed as a function of the variables $r, \vartheta, z$, we have

$$L(r, \vartheta, z, \dot{r}, \dot{\vartheta}, \dot{z}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2) - V(r, \vartheta, z)$$

and the Lagrange equations turn out to be

$$m\ddot{r} - mr\dot{\vartheta}^2 + \frac{\partial V}{\partial r} = 0 \ , \qquad mr^2\ddot{\vartheta} + 2mr\dot{r}\dot{\vartheta} + \frac{\partial V}{\partial \vartheta} = 0 \ , \qquad m\ddot{z} + \frac{\partial V}{\partial z} = 0 \ .$$

The conjugate momenta turn out to be $p_r = m\dot{r}$ (the radial component of the linear momentum), $p_\vartheta = mr^2\dot{\vartheta}^2$ (the $z$ component of the angular momentum), $p_z = m\dot{z}$ (the $z$ component of the linear momentum).

Example Point mass in spherical coordinates $\rho, \vartheta, \varphi$.

The kinetic energy $K$ has the expression (2.3.6); if $V(\rho, \vartheta, \varphi)$ denotes the potential energy, the Lagrangian is $L = K - V$ and the Lagrange equations are

$$m\ddot{\rho} - m\rho\dot{\vartheta}^2 - m\rho\sin^2\vartheta\dot{\varphi}^2 + \frac{\partial V}{\partial \rho} = 0 \ , \qquad m\rho^2\ddot{\vartheta} + 2m\rho\dot{\rho}\dot{\vartheta} - 2m\rho^2\sin\vartheta\cos\vartheta\dot{\varphi}^2 + \frac{\partial V}{\partial \vartheta} = 0$$
$$m\rho^2\sin^2\vartheta\ddot{\varphi} + 2m\rho\sin^2\vartheta\dot{\rho}\dot{\varphi} + 2m\rho^2\sin\vartheta\cos\vartheta\dot{\vartheta}\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 \ .$$

It is easy to see that $p_\rho$ is the radial component of the linear momentum, $p_\varphi$ is the $z$ component of the angular momentum, while $p_\vartheta$ is the modulus of the projection of the angular momentum on the $xy$ plane.

In these examples the kinetic energy does not contain mixed terms in the velocities (the kinetic matrix is diagonal). Correspondingly, each Lagrange equation contanis only one second derivative.

Example Point mass in a plane, described in non orthogonal Cartesian coordinates $\xi, \eta$.

From expression (2.3.7) of $K$, if $V(\xi, \eta)$ is the potential energy, the Lagrangian is $L = K - V$ and the Lagrange equations are

$$m\ddot{\xi} + m\cos\alpha\ddot{\eta} + \frac{\partial V}{\partial \xi} = 0 \ , \qquad m\cos\alpha\ddot{\xi} + m\sin^2\alpha\ddot{\eta} + \frac{\partial V}{\partial \eta} = 0 \ .$$

Other examples of Lagrange functions and Lagrange equations can be found in the exercise part of these lecture notes.

### 2.4.3 General Lagrangian systems

The Lagrange equations (2.4.7) have been here deduced in connection with mechanical problems: more precisely, we considered a holonomic system with $n$ degrees of freedom, possibly subject to ideal constraints, which moves under the effect of an assigned system of active forces; in such a context, starting from the Newton equations, we deduced equations (2.4.7). Equations of this form, however, are interesting far beyond mechanics: as we shall see below, in section 2.8 devoted to the so-called variational calculus, equations of the form (2.4.7) appear naturally in problems of geometry, or optics, and in general in problems of optimization, when the variable on which we optimize is not a number, but a function. In general however, for problems that go beyond mechanics, the Lagrange function $L$ does not have the form $L = K - V$, with $K$ of the form (2.3.2), but can be any function of the $n$ unknowns, of their velocities and of the independent variable (of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $t$, in the mechanical case). The Lagrangian systems for which it is $L = K - V$, with $K$ of the form (2.3.2), will be called *mechanical,* or also *natural* Lagrangian systems, while the other ones will be called *general* Lagrangian systems. In the next subsections we shall mainly deal with mechanical systems, but occasionally we shall also comment the general case, putting in evidence those properties of the Lagrange equations which are valid in the general case, too.

### 2.4.4 The normal form of Lagrange equations

As seen so far, the Lagrange equations, in their form (2.4.2) or (2.4.7), are equalities that are satisfied during the motion. Here we make a step forward, showing that in the mechanical case equations (2.4.2), and consequently equations (2.4.7), have the form of second order differential equations in the unknowns $q_1(t), \ldots, q_n(t)$ that, thanks to the properties of the kinetic matrix, can

always be put in normal form. This is fundamental, because the Cauchy theorem of existence and uniqueness of solutions then holds: initial data determine the movement (and consequently the constraint reactions), and so the Lagrange equation (2.4.2) or (2.4.7) may well be said to be the equations of motion of the system. Precisely:

PROPOSITION 24 *In the mechanical case, the Lagrange equations (2.4.2) are equivalent to a system of $n$ second order differential equations of the form*

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) \ , \tag{2.4.10}$$

$\mathbf{f}$ *being given by*

$$\mathbf{f} = a^{-1}(\mathbf{Q} - \mathbf{g}) \ , \tag{2.4.11}$$

*where*

$$g_h = \sum_{jk} \Big( \frac{\partial a_{hk}}{\partial q_j} - \frac{1}{2} \frac{\partial a_{jk}}{\partial q_h} \Big) \dot{q}_j \dot{q}_k + \sum_k \Big( \frac{\partial b_h}{\partial q_k} - \frac{\partial b_k}{\partial q_h} + \frac{\partial a_{hk}}{\partial t} \Big) \dot{q}_k + \frac{\partial b_h}{\partial t} - \frac{1}{2} \frac{\partial c}{\partial q_h} \ ; \tag{2.4.12}$$

*for any assigned initial datum $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$, both the motion $\mathbf{q}(t)$ and the constraint reactions $\mathbf{\Phi}_1(t), \dots, \mathbf{\Phi}_N(t)$, satisfying the ideality condition (2.2.10), are uniquely determined.*

The precise expression of $g_h$, added here for completeness, will not be much relevant for us: its only essential property is that it includes $\mathbf{q}, \dot{\mathbf{q}}$ and $t$, but not the second derivaties $\ddot{\mathbf{q}}$ of the unknown functions. The second order system (2.4.10) can always be rewritten as a system of $2n$ equations of first order, of the form

$$\dot{\mathbf{q}} = \mathbf{v} \ , \qquad \dot{\mathbf{v}} = \mathbf{f}(\mathbf{q}, \mathbf{v}, t) \ . \tag{2.4.13}$$

PROOF. From expression (2.3.2) of $K$, thanks to the symmetry of the kinetic matrix, it follows

$$\frac{\partial K}{\partial \dot{q}_h} = \sum_{k=1}^{n} a_{hk}(\mathbf{q}, t) \dot{q}_k + b_h(\mathbf{q}, t) \ ;$$

it is then straightforward to check that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} = \sum_{k=1}^{n} a_{hk}(\mathbf{q}, t) \ddot{q}_k + g_h(\mathbf{q}, \dot{\mathbf{q}}, t) \ ,$$

$g_h(\mathbf{q}, \dot{\mathbf{q}}, t)$ being a convenient function of $\mathbf{q}$, $\dot{\mathbf{q}}$, $t$ but not of $\ddot{\mathbf{q}}$. (In other words: the terms containing the second derivatives $\ddot{q}_h$ come only from $K_2$, and appear in the above form.) An elementary computation shows $g_h$ has the expression (2.4.12). The Lagrange equations appear then in the form

$$a(\mathbf{q}, t) \ddot{\mathbf{q}} = Q(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) \ ;$$

from this expression, thanks to the invertibility of the kinetic matrix, equation (2.4.10) follows, $\mathbf{f}$ being as in (2.4.11). The uniqueness of the motion for given initial data is a consequence of the Cauchy theorem of existence and uniqueness of solutions of differential equations. The constraint reactions $\mathbf{\Phi}_i$ exerted on each point mass $P_i$, $i = 1, \dots, N$, are in turn determined by the Newton equations, $\mathbf{\Phi}_i = m_i \mathbf{a}_i - \mathbf{F}_i$. □

- Using the conjugate momenta $p_h$, it is also spontaneous to write the $n$ Lagrange equations as a system of $2n$ equations of the form

$$p_h = \frac{\partial L}{\partial \dot{q}_h}(\mathbf{q}, \dot{\mathbf{q}}, t) \ , \qquad \dot{p}_h = \frac{\partial L}{\partial q_h}(\mathbf{q}, \dot{\mathbf{q}}, t) \ , \qquad h = 1, \dots, n$$

(on the left we have the very definition of $p_h$, on the right the Lagrange equation). *The left equations, however, are not in normal form,* because they are not solved with respect to the $\dot{q}_h$. An alternative way to obtain from the Lagrange equations a system of $2n$ equations in normal form, comes then spontaneously: namely inverting the first $n$ equations, making explicit $\dot{\mathbf{q}}$ as a function of $\mathbf{q}$, $\mathbf{p}$ and $t$, and then replacing $\dot{\mathbf{q}}$ in the second group of equations. In this way, we obtain a system of $2n$ equations in which the fundamental variables are not positions and velocities, but positions and momenta. A new formulation of mechanics, alternative to Lagrangian mechanics and called *Hamiltonian mechanics,* starts from here. Such an alternative formulation is very interesting and fruitful, but we haven't the possibility to enter it in these notes.

EXERCISE 32 *Put in normal form the Lagrange equations appearing in the examples of section 2.4.2. Pay special attention to the last of them.*

Note that, thanks to (2.2.4), there is a one-to-one correspondence between the initial datum $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ and the initial datum $(\mathbf{w}_0, \dot{\mathbf{w}}_0)$, or equivalently $(P_{0,1}, \dots, P_{0,N}, \mathbf{v}_{0,1}, \dots, \mathbf{v}_{0,N})$, if we impose the latter is compatible with the constraints, that is $\mathbf{w}_0$ stays on the constraint manifold and $\dot{\mathbf{w}}_0$ is tangent to it. Keeping this in mind, we may also reformulate Proposition 24 in the following way:

PROPOSITION 25 *Consider a holonomic system of $N$ point masses with $n$ degrees of freedom, subject to given active forces $\mathbf{F}_i$, $i = 1, \dots, N$; assigne initial positions and velocities, compatible with the constraints. There exists an unique choice of the constraint reactions $\mathbf{\Phi}_i$, satisfying the ideality condition (2.2.7), such that the motion of the points, subject to the Newton equations $m_i \mathbf{a}_i = \mathbf{F}_i + \mathbf{\Phi}_i$, is compatible with the constraints. Such a motion satisfies the Lagrange equations (2.4.2), and is determined by them.*

It is immediate to check that for general Lagrangian systems, in any case, independently of the form of $L$, equations (2.4.7) are a system of $n$ second order differential equations in the variables $q_1(t), \dots, q_n(t)$, linear in $\ddot{q}_1, \dots, \ddot{q}_n$, of the form

$$\sum_{k=1}^{n} \frac{\partial^2 L}{\partial \dot{q}_h \partial \dot{q}_k} \ \ddot{q}_k + g_h = 0 \ , \qquad h = 1, \dots, n \ ,$$

with some $g_h = g_h(\mathbf{q}, \dot{\mathbf{q}}, t)$. Quite clearly, the condition for the system can be put in normal form is that the Hessian of $L$ with respect to the $\dot{q}_h$ does not vanish:

$$\det \Big( \frac{\partial^2 L}{\partial \dot{q}_h \partial \dot{q}_k} \Big) \ \neq \ 0 \ .$$

For natural systems the matrix $\big(\frac{\partial^2 L}{\partial \dot{q}_h \partial \dot{q}_k}\big)$ is nothing but the kinetic matrix $a = (a_{hk})$.

### 2.4.5 Lagrange equations and rigid bodies

As already commented in section 2.2.3-B, the rigidity constraint, in the usual assumptions such that the cardinal equations hold, is an ideal constraint; a quite relevant consequence is that for such systems the Lagrange equations do hold. In fact, it is possible to prove (see appendix B) that *for rigid systems, cardinal equations and Lagrange equations are equivalent.*

An obvious generalization is the case of rigid bodies with further constraints (for example, rigid bodies with a fixed point, or a fixed axis), assuming of course the additional constraints do not violate the ideality condition. Similarly obvious is the extension to systems including several rigid bodies, possibly constrained to each other by hinges or other devices, of course in the assumption that all constraints are ideal. All of this, it is understood, limited to the case of rigid bodies consisting of a finite number of points. Quite delicate is instead the extension to rigid bodies consisting of infinitely many elements, in particular continuous rigid bodies. The question itself is nontrivial, because (already within Newtonian mechanics) it is not that easy to understand what does it mean dynamics of infinitely many points subject to infinitely many constraints.[14] We shall not go further into such a delicate problem, and limit ourselves to a few comments.

- First of all, the kinematics of rigid bodies, as holonomic systems with six degrees of freedom, is totally independent of the number of points of the system (the position of each point, no matter if they are infinite, is in any case determined as a function of six free coordinates).

- The dynamical quantities entering the Lagrangian, namely $K$ and $V$, are well defined, in analogy to finite systems, by simply replacing sums by integrals (precisely as we do in Newtonian mechanics, when we introduce the quantities $\mathbf{Q}$, $\mathbf{M}$, $\mathbf{R}^{\text{ext}}$ and $\mathbf{N}^{\text{ext}}$, entering the cardinal equations). This means the Lagrange equations, intended as differential equations in six variables, by themselves, regardless of whether they are appropriate or not, are well defined, and coincide with those of finite bodies with the same $K$ and $V$ as functions of the free coordinates (similar considerations hold for the cardinal equations, too).

- *It is not easy to say anything else.* We may imagine a continuous rigid body to be the limit of a sequence of finite systems, each with the same $K$ and $V$ and thus the same Lagrange equations, regardless of the number of points (the same $\mathbf{Q}$, $\mathbf{M}$, $\mathbf{R}^{\text{ext}}$, $\mathbf{N}^{\text{ext}}$, and thus the same cardinal equations, in the Newtonian framework). This makes *highly reasonable* to *assume* the dynamics of continuous rigid bodies is described by the Lagrange equations (or by the cardinal equations). But in any case, *it is an assumption:* logically, it is not possible to deduce (in the mathematical sense) the behavior of an infinite system, in particular a continuous one, from that of finite systems.

In the following, we shall assume the Lagrange equations also hold for rigid bodies consisting of infinitely many points, in particular for continuous rigid bodies.

It is an easy exercise to check, in particular cases, that the Lagrange equations are equivalent to the cardinal equations.

Example Consider a rigid body which is free of rotate around a fixed axis. With obvious notation, it is then $K = \frac{1}{2} I \dot{\varphi}^2$, while the potential energy is some function $V(\varphi)$ such that $N = -\frac{\mathrm{d}V}{\mathrm{d}\varphi}$, where

[14] This is clearly a general problem concerning the dynamics of continuous bodies, no matter if rigid or not, and its possible "deduction" from the dynamics of systems with finitely many point masses.

$N$ is the resultant momentum of the external forces with respect the rotation axis. From the Lagrangian $L = K - V$ we then obtain the Lagrange equation

$$I\ddot{\varphi} - N = 0 \ ,$$

and this is the cardinal equation for the angular momentum, projected on the rotation axis.

EXAMPLE Consider a homogeneous disc which rolls without slipping on straight guide. In absence of constraint, we would have (with obvious meaning of the symbols, $I$ referring to the central axis) $K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\varphi}^2$, with some $V = V(x, \varphi)$. If $F$ denotes the $x$ component of the resultant of external forces, and $N$ is the resultant external momentum with respect to the central axis, it is $F = -\frac{\partial V}{\partial x}$, $N = -\frac{\partial V}{\partial \varphi}$. If we now introduce the constraint of pure rolling $x = x_0 - R\varphi$, we obtain $L = \frac{1}{2}(mR^2 + I)\dot{\varphi}^2 - V(x_0 - R\varphi, \varphi)$. In the corresponding Lagrange equation

$$(mR^2 + I)\ddot{\varphi} + RF - N = 0$$

we immediately recognize the cardinal equation for the angular momentum, relative to any point of the guide, projected in the direction orthogonal to the disc.

### 2.4.6 Invariance properties of the Lagrange equations

*A. Invariance in form of the equations.* The procedure itself we used to deduce the Lagrange equations for a system of point masses, with arbitrary choice of the free coordinates $(q_1, \dots, q_n)$, allows us to conclude that the equations of motion always have the form (2.4.7), or more in general (2.4.2), in any system of coordinates, that is to say, the Lagrange equations are *invariant in form* for change of coordinates. Such a property, however, is independent of the mechanical context in which we deduced the equations: it holds for general Lagrangian systems, and directly follows from the form of the equations, which by itself is invariant under change of coordinates.

Indeed, let us consider any change of coordinates (possibly dependent of time), from coordinates $(q_1, \dots, q_n)$ to new coordinates $(\tilde{q}_1, \dots, \tilde{q}_n)$, that is a regular map

$$q_h = q_h(\tilde{q}_1, \dots, \tilde{q}_n, t) \ , \qquad h = 1, \dots, n \ , \tag{2.4.14}$$

such that its Jacobian matrix $J$ has nonvanishing determinant:

$$\det J \neq 0 \ , \qquad \text{where} \quad J_{hk} = \Big(\frac{\partial q_h}{\partial \tilde{q}_k}\Big) \ ; \tag{2.4.15}$$

such a conditions ensures indeed that the transformation is invertible at least locally, and has regular inverse (is a *local diffeomorphism*).[15] From (2.4.14) we obtain, by derivation, the extension of the transformation to the velocities:

$$\dot{q}_h(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) = \sum_{k=1}^{n} \frac{\partial q_h}{\partial \tilde{q}_k}\dot{\tilde{q}}_k + \frac{\partial q_h}{\partial t} \ . \tag{2.4.16}$$

[15]Equation (2.4.15) also guarantees that the matrix $\frac{\partial \tilde{\mathbf{w}}}{\partial \tilde{\mathbf{q}}}$, where we posed $\tilde{\mathbf{w}}(\tilde{\mathbf{q}}, t) = \mathbf{w}(\mathbf{q}(\tilde{\mathbf{q}}, t), t)$, has rank $n$, and consequently $(\tilde{q}_1, \dots, \tilde{q}_n)$ are good free coordinates. We are here putting in evidence the expression of the old coordinates as functions of the new ones, which is concretely used to substitute variables inside a function. But it would be equivalent, thanks to the invertibility of the transformation, writing the new coordinates as functions of the old ones, $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}, t)$.

By replacing the variables inside $L$, we obtain the new Lagrangian

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) = L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) \ , \tag{2.4.17}$$

and it is not difficult to prove, in complete generality, that the movement, in the new variables, satisfies the Lagrange equations associated to $\tilde{L}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} - \frac{\partial \tilde{L}}{\partial \tilde{q}_h} = 0 \ , \qquad h = 1, \dots, n \ . \tag{2.4.18}$$

Precisely:

Proposition 26 *Consider a Lagrangian system of Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$; introduce a change of variables (2.4.14) satisfying (2.4.15), and let (2.4.16) be its extension to velocities. Finally, let $\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t)$ be the Lagrangian obtained from $L$ by replacement of variables, according to (2.4.17). The movement $\tilde{\mathbf{q}}(t)$ is a solution of the Lagrange equations associated to $\tilde{L}$, if and only if the corresponding movement $\mathbf{q}(t)$, image of $\tilde{\mathbf{q}}(t)$ through (2.4.14), is a solution of the Lagrange equations associated to $L$.*

Proof. From the expression of $\tilde{L}$, taking the derivatives and using the easy relations

$$\frac{\partial \dot{q}_k}{\partial \dot{\tilde{q}}_h} = \frac{\partial q_k}{\partial \tilde{q}_h} \ , \qquad \frac{\partial \dot{q}_k}{\partial \tilde{q}_h} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial q_k}{\partial \tilde{q}_h}$$

(the former immediately follows from (2.4.16), the latter is obtained by exchanging partial and total derivative, according to Lemma 22), we obtain

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} &= \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\frac{\partial q_k}{\partial \tilde{q}_h} \ , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} = \sum_{k=1}^{n} \Big[ \Big( \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} \Big) \frac{\partial q_k}{\partial \tilde{q}_h} + \frac{\partial L}{\partial \dot{q}_k} \Big( \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial q_k}{\partial \tilde{q}_h} \Big) \Big] \ , \\ \frac{\partial \tilde{L}}{\partial \tilde{q}_h} &= \sum_{k=1}^{n} \Big[ \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial \tilde{q}_h} + \frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial q_k}{\partial \tilde{q}_h} \Big] \ . \end{aligned}$$

As a consequence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_h} - \frac{\partial \tilde{L}}{\partial \tilde{q}_h} = \sum_{k=1}^{n} J_{kh} \Big[ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \Big] \ ,$$

and thanks to (2.4.15), we conclude that the Lagrange equations relative to $\tilde{L}$ are satisfied, if and only if the Lagrange equations relative to $L$ are satisfied. □

Later on, after introducing the variational principles, we shall see a much simpler and natural proof of this proposition, which in that context will appear even obvious.

- From the invariance in form of the Lagrange equations under arbitrary changes of coordinates, it follows that if the motion of a system obeys the Lagrange equations in a given system of coordinates, then it obeys the Lagrange equations in any other system of coordinates. Now, it is trivial to verify that the Newton equations for a system of $N$ point masses (without constraints), written in the usual Cartesian coordinates, are precisely the Lagrange equations relative to $L = K - V$, with the usual definitions of $K$ and $V$; as a consequence, they

mantain the form of Lagrange equations, relative to the Lagrangian $\tilde{L}$ obtained from $L$ by substitution of variables, in any other system of coordinates. This is a particularly simple and direct deduction of the Lagrange equations, for systems of unconstrained point masses described in arbitrary coordinates.

*B. The gauge invariance.* Quite evidently, different Lagrange functions may lead to the same Lagrange equations: for example, if we add to $L$ a constant, or multiply $L$ by a constant, it is evident that the equations do not change. Beyond such trivial cases, it is not difficult to check that the Lagrange equations do not change, if two Lagrangians $L$ and $L'$ differ by a function $L_0(\mathbf{q}, \dot{\mathbf{q}}, t)$, which is the total derivative with respect to time of any function $F(\mathbf{q}, t)$:

$$L_0(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{\mathrm{d}F}{\mathrm{d}t}(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{k=1}^{n} \frac{\partial F}{\partial q_k}(\mathbf{q}, t)\dot{q}_k + \frac{\partial F}{\partial t}(\mathbf{q}, t) \ . \tag{2.4.19}$$

Once more, such a property follows directly from the form of the Lagrange equations, and so it holds for general Lagrangian systems, beyond the mechanical case:

PROPOSITION 27 *For any choice of the function $F(\mathbf{q}, t)$ and of the real constant $c \neq 0$, the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and the Lagrangian*

$$L'(\mathbf{q}, \dot{\mathbf{q}}, t) = c\, L(\mathbf{q}, \dot{\mathbf{q}}, t) + L_0(\mathbf{q}, \dot{\mathbf{q}}, t) \ , \qquad \textit{with} \qquad L_0 = \frac{\mathrm{d}F}{\mathrm{d}t} \ ,$$

*lead to the same Lagrange equations.*

PROOF. It is enough to show that the term $L_0 = \frac{\mathrm{d}F}{\mathrm{d}t}$ does not contribute to the equations of motion, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \dot{q}_h} - \frac{\partial L_0}{\partial q_h} = 0 \ .$$

This is immediate since, from expression (2.4.19) of $L_0$, also using Lemma 22, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \dot{q}_h} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial q_h} = \frac{\partial}{\partial q_h}\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial L_0}{\partial q_h} \ .$$

□

Such a property of invariance of the Lagrange equations, by adding to the Lagrangian a term of the special form $\frac{\mathrm{d}F}{\mathrm{d}t}$, is called *gauge invariance*.[16] This property too will be better understood, and appear natural, in the context of variational principles.

## 2.5 Potentials dependent on velocity

In the physical world, there are two simple examples of forces which depend on the velocity, but haven't dissipative nature, namely the Coriolis force

$$\mathbf{F} = -2m\,\boldsymbol{\omega} \times \dot{\mathbf{q}} \ , \tag{2.5.1}$$

[16]The expression *gauge invariance* comes from electrodynamics, where it is used to characterize the analogous fact that by adding to the vector potential $\mathbf{A}$ the gradient $\nabla F$ of any scalar function, the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ does not change.

which is introduced (similarly to the centifugal force) to describe a point mass in a rotating system with instantaneous angular velocity $\boldsymbol{\omega}$, so as we can proceed as if the system were inertial, and the Lorentz force

$$\mathbf{F} = e(\mathbf{E} - \mathbf{B} \times \dot{\mathbf{q}}) \tag{2.5.2}$$

wich determines the motion of an electric charge $e$ in a given electromagnetic field. The problem then arises to appropriately treat similar forces within the Lagrangian formalism. A possible way is, of course, writing the Lagrange equations in the improper form (2.4.2), or in the mixed form (2.4.8), that is treating such velocity dependent forces similarly to dissipative forces. But we can do better, namely we can write the Lagrange equations in the much preferable proper form (2.4.7), if we accept the presence, in the Lagrangian, of (simple) potential terms which are not positional, but depend on the velocity too, actually are linear in the velocity. The idea is simple: from the improper form of the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} = Q_h$$

we worked out the proper form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = 0 \ ,$$

with $L = K - V$, assuming $Q_h$ was positional and deducible from some function $V(\mathbf{q})$ via $Q_h = -\frac{\partial V}{\partial \mathbf{q}}$. It is clear however that the proper form of the equations also follows if, more generally, there exists a function $V(\mathbf{q}, \dot{\mathbf{q}})$, such that

$$Q_h = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{q}_h} - \frac{\partial V}{\partial q_h} \ . \tag{2.5.3}$$

It is not difficult to check the Coriolis force and the Lorentz force fall into this case.

### 2.5.1 The Coriolis force

Let us assume here for simplicity the angular velocity $\boldsymbol{\omega}$ is constant; in such a case, the so-called fictitious forces (or inertial forces) we need to introduce for each point mass $P$ of the system, in addition to the "real" forces, so as we can work in the rotating system as if it were inertial, reduce to the centrifugal force $\mathbf{F}_c = m\omega^2 r \mathbf{e}_r$, where $r$ denotes the distance of $P$ from the rotation axis and $\mathbf{e}_r$ is the radial unit vector of the cylindrical coordinates, and the Coriolis force (2.5.1). The centrifugal force is a positional conservative force, described by the potential $V_c = -\frac{1}{2}m\omega^2 r^2$, which sums to the potential of the other possibly present positional conservative forces. Concerning instead the Coriolis force, the following easy proposition holds:

Proposition 28 *In the case of constant angular velocity, the Coriolis force (2.5.1) deduces, through (2.5.3), from a potential $V_1(\mathbf{q}, \dot{\mathbf{q}})$, linear in $\dot{\mathbf{q}}$, given by*

$$V_1(\mathbf{q}, \dot{\mathbf{q}}) = m\, \boldsymbol{\omega} \times \dot{\mathbf{q}} \cdot \mathbf{q} \ . \tag{2.5.4}$$

Proof. From the expression of $V_1$ we obtain immediately

$$\frac{\partial V_1}{\partial q_h} = m(\boldsymbol{\omega} \times \dot{\mathbf{q}})_h \ ;$$

moreover (observing that, by a well known property of the mixed vector product, it is $V_1 = m\,\mathbf{q} \times \boldsymbol{\omega} \cdot \dot{\mathbf{q}}$), we also have

$$\frac{\partial V_1}{\partial \dot{q}_h} = m(\mathbf{q} \times \boldsymbol{\omega})_h \ , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V_1}{\partial \dot{q}_h} = m(\dot{\mathbf{q}} \times \boldsymbol{\omega})_h \ .$$

The conclusion is immediate. □

The Lagrangian of the system is then

$$L = K - V_0 - V_1 \ , \tag{2.5.5}$$

where $K = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$ as it appears in the rotating frame, while $V_0$ (the subscript zero reminds the term is of degree zero in the velocities) includes the potential of the centrifugal force and of the other possibly present active forces.

Exercise 33 *Verify that the Lagrangian (2.5.5), with $V_1$ of the form (2.5.4), is the Lagrangian of a point mass in a rotating frame, also in the case of non constant $\omega$. (More precisely: if $\boldsymbol{\omega}$ depends on time, the term $V_1$ produces an additional term $-m\,\dot{\boldsymbol{\omega}} \times \mathbf{q}$, which effectively enters, in such a case, the fictitious force.)*

Working in a non inertial reference frame, proceeding as if it were inertial through the expedient of introducing convenient fictitious forces, is spontaneous within Newtonian mechanics, and can also be done, as we have just seen, in the Lagrangian context. But the most natural way to proceed, within the Lagrangian formalism, is different: without necessity of invoking the nontrivial theory of relative motions, we can easily pass from an inertial frame to a rotating one (or to any other accelerated frame) by simply writing a change of coordinates depending on time. Suppose we are dealing, as above, with a frame which rotates uniformly, with angular velocity $\omega$, around the $z$ axis. If $x$, $y$, $z$ and $q_1$, $q_2$, $q_3$ denote, respectively, the Cartesian coordinates of $P$ in the inertial frame and in the rotating frame, then the change of coordinates is given by

$$x = q_1 \cos\omega t - q_2 \sin\omega t \ , \qquad y = q_1 \sin\omega t + q_2 \cos\omega t \ , \qquad z = q_3 \ , \tag{2.5.6}$$

with correspondingly

$$\begin{aligned} \dot{x} &= \dot{q}_1 \cos\omega t - \dot{q}_2 \sin\omega t - \omega(q_1 \sin\omega t + q_2 \cos\omega t) \\ \dot{y} &= \dot{q}_1 \sin\omega t + \dot{q}_2 \cos\omega t + \omega(q_1 \cos\omega t - q_2 \sin\omega t) \\ \dot{z} &= \dot{q}_3 \ . \end{aligned}$$

In the inertial frame the kinetic energy is $\tilde{K} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$; developing the squares and substituting, we find $\tilde{K} = K_2 + K_1 + K_0$, with

$$\begin{aligned} K_2 &= \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \\ K_1 &= m\omega(q_1\dot{q}_2 - q_2\dot{q}_1) = m\,\boldsymbol{\omega} \cdot \mathbf{q} \times \dot{\mathbf{q}} = -m\,\boldsymbol{\omega} \times \dot{\mathbf{q}} \cdot \mathbf{q} \\ K_0 &= \frac{1}{2}m\omega^2(q_1^2 + q_2^2) \ . \end{aligned}$$

It clearly appears that the term $K_2$ is the kinetic energy in the rotating system, while the kinetic term $K_1$ is the opposite of the potential term $V_1$ which produces the Coriolis force, and finally $K_0 = -V_c$. So, *the Lagrangian is the same as (2.5.5).* It is interesting to observe that the same terms of the Lagrangian appear in one case as kinetic, in the other case as potential. In other examples, if we proceed in the most spontaneous way, we obtain Lagrangians which do not coincide, but differ by a term of the form $L_0 = \frac{\mathrm{d}}{\mathrm{d}t}F$ (with suitable $F(\mathbf{q}, t)$), and so are equivalent.

EXERCISE 34 *Study the motion of a simple pendulum, whose oscillation plane rotates uniformly around the vertical (centrifugal pendulum); work both in the inertial frame and in the rotating frame, and compare the results. In a similar way, study the motion of a pendulum, whose suspension point $C$ moves along the vertical axis $y$ with some assigned law $y_C = f(t)$.* [The problem is treated in detail in the exercise part of these notes.]

### 2.5.2 The Lorentz force

Let us come to the Lorentz force (2.5.2). In the case of constant uniform magnetic field, we do not need to do anything: the magnetic part of the Lorentz force gets *identical* to the Coriolis force, if only we identify $e\mathbf{B}$ with $2m\boldsymbol{\omega}$, and so it comes from $V_1 = \frac{1}{2}e\,\mathbf{B}\times\dot{\mathbf{q}}\cdot\mathbf{q}$, while the electric part comes from a suitable potential $\Phi$ through $\mathbf{E} = -e\nabla\Phi$.

But without much difficulty, we can treat the case of *any* assigned electromagnetic field $\mathbf{E}(\mathbf{q},t)$, $\mathbf{B}(\mathbf{q},t)$, by simply introducing, besides the "scalar potential" $\Phi$, the "vector potential" $\mathbf{A}$; in electrodynamics, we recall, $\mathbf{E}$ and $\mathbf{B}$ are related to $\Phi$ and $\mathbf{A}$ through

$$\mathbf{E} = -\Big(\nabla\Phi + \frac{\partial\mathbf{A}}{\partial t}\Big)\,, \qquad \mathbf{B} = \nabla\times\mathbf{A}\,. \tag{2.5.7}$$

Indeed, the following proposition holds:

PROPOSITION 29 *The Lorentz force (2.5.2) is deduced, through (2.5.3), from the potential*

$$V(\mathbf{q},\dot{\mathbf{q}}) = e\,\Phi - e\,\dot{\mathbf{q}}\cdot\mathbf{A}\,. \tag{2.5.8}$$

The vector $\mathbf{j} = e\dot{\mathbf{q}}$ is interpreted as the *current* associeted to the motion of the charge $e$ with velocity $\dot{\mathbf{q}}$; so, precisely as the electric charge couples with the scalar potential $\Phi$, the current couples with the vector potential $\mathbf{A}$ (and it gets spontaneous to rewrite (2.5.8) in the form of a scalar product of four-components vectors, as in relativity theory).

PROOF. Using potentials, the Lorentz force assumes the form

$$\mathbf{F} = -e\Big(\nabla\Phi + \frac{\partial\mathbf{A}}{\partial t}\Big) + e\,\dot{\mathbf{q}}\times\nabla\times\mathbf{A}\,;$$

it is then enough to show that the r.h.s. of this expression coincides with $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial\dot{\mathbf{q}}} - \frac{\partial V}{\partial\mathbf{q}}$, with $V$ is as in (2.5.8). This is not difficult: by deriving we get

$$\frac{\partial V}{\partial\dot{q}_h} = -eA_h\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial\dot{q}_h} = -e\Big(\frac{\partial A_h}{\partial t} + \sum_{k=1}^{3}\dot{q}_k\frac{\partial A_h}{\partial q_k}\Big)\,, \qquad \frac{\partial V}{\partial q_h} = e\Big(\frac{\partial\Phi}{\partial q_h} - \sum_{k=1}^{3}\dot{q}_k\frac{\partial A_k}{\partial q_h}\Big)\,,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial\dot{q}_h} - \frac{\partial V}{\partial q_h} = -e\Big(\frac{\partial\Phi}{\partial q_h} + \frac{\partial A_h}{\partial t}\Big) + e\sum_{k=1}^{3}\dot{q}_k\Big(\frac{\partial A_k}{\partial q_h} - \frac{\partial A_h}{\partial q_k}\Big)\,;$$

it is then easy to check that[17]

$$\sum_{k=1}^{3}\dot{q}_k\Big(\frac{\partial A_k}{\partial q_h} - \frac{\partial A_h}{\partial q_k}\Big) = (\dot{\mathbf{q}}\times\nabla\times\mathbf{A})_h\,,$$

[17]Such an equality is easily remembered in the form $\mathbf{v}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}$, $\mathbf{v} = \dot{\mathbf{q}}$, very similar to the common formula for the double vector product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$.

and this concludes the proof. □

So, the Lagrangian of a particle of mass $m$ and electric charge $e$ in a given electromagnetic field is

$$L = K - V = \frac{1}{2} m \dot{\mathbf{q}}^2 + e \dot{\mathbf{q}} \cdot \mathbf{A} - e\Phi \ .$$

## 2.6 The conservation laws in Lagrangian mechanics

*Foreward:* in the previous sections we deduced the Lagrange equations and discussed their most relevant properties. Within the mechanical, or natural, context, we first met the "improper form" (2.4.2) of the equations, where only the kinetic energy $K = K_2 + K_1 + K_0$ does appear, and then their "proper form" (2.4.7), where the Lagrangian $L = K - V$ enters. To include in our discussion the Coriolis force and the Lorentz force, we added the possibility the potential is not positional, $V(\mathbf{q}, \dot{\mathbf{q}})$. Finally, we mentioned the "general" Lagrangian systems, with any Lagrange function $L(\mathbf{q}, \dot{\mathbf{q}})$. In the next two sections we are primarily interested in natural Lagangian systems, but it will also be useful, for a better comprehension, to understand the level of generality of the different statements. Among natural systems, a special role is played by the case, actually the simplest and most common one, in which the kinetic energy has only the quadratic term, $K(\mathbf{q}, \dot{\mathbf{q}}) = K_2(\mathbf{q}, \dot{\mathbf{q}})$: this is indeed the case of mechanical systems with fixed constraints and time independent coordinates, to which we shall devote a special attention.

### 2.6.1 The conservation of energy

Consider any Lagrangian system with $n$ degrees of freedom, with Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, and introduce the function

$$E(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{h=1}^{n} \dot{q}_h \frac{\partial L}{\partial \dot{q}_h}(\mathbf{q}, \dot{\mathbf{q}}, t) - L(\mathbf{q}, \dot{\mathbf{q}}, t) \ . \tag{2.6.1}$$

On the basis of the Lagrange equations, without restrictions on the form of $L$ (and so for general Lagrangian systems, too), it is easy to compute the total derivative of $E(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)$ with respect to $t$:

$$\begin{aligned} \dot{E} &= \sum_{h=1}^{n} \Big( \ddot{q}_h \frac{\partial L}{\partial \dot{q}_h} + \dot{q}_h \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_h} \Big) - \sum_{h=1}^{n} \Big( \dot{q}_h \frac{\partial L}{\partial q_h} + \ddot{q}_h \frac{\partial L}{\partial \dot{q}_h} \Big) - \frac{\partial L}{\partial t} \\ &= \sum_{h=1}^{n} \dot{q}_h \Big( \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} \Big) - \frac{\partial L}{\partial t} \\ &= -\frac{\partial L}{\partial t} \ . \end{aligned}$$

So, in the case $L$ does not depend explicitly on time, we see that $E$ is a constant of motion.

If in addition the Lagrangian has the form $L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$, with $K(\mathbf{q}, \dot{\mathbf{q}}) = K_2(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{h,k} a_{hk}(\mathbf{q}) \dot{q}_h \dot{q}_k$ (natural systems with fixed constraints and pure positional conservative forces), then the function $E$ is easily interpreted: indeed it is not difficult to see that

$$E = 2K - (K - V) = K + V \ , \tag{2.6.2}$$

and so $E$ is the energy of the system, written as a function of the Lagrangian coordinates $\mathbf{q}$ and $\dot{\mathbf{q}}$.

Equality (2.6.2) can be verified directly, but the best idea is to use the Euler lemma on homogeneus functions.

DEFINITION 13 *The function $f(x_1, \ldots, x_n)$ is said to be* homogeneous of degree $\alpha$, *if for any $\lambda > 0$ and any choice of $x_1, \ldots, x_n$ it is*

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^\alpha f(x_1, \ldots, x_n) \ . \tag{2.6.3}$$

Any homogeneous polynomial is evidently a homogeneous function of the same degree as the polynomial; $f(x_1, x_2) = \sqrt{x_1 + x_2^2/x_1}$ is homogeneous of degree $\alpha = 1/2$.

LEMMA 30 *If $f$ is homogeneous of degree $\alpha$, then*

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = \alpha f \ .$$

PROOF. Derive equality (2.6.3) with respect to $\lambda$, and then set $\lambda = 1$. □

To deduce (2.6.2), it is enough to apply the Euler lemma to $K = K_2$.

Consider now the case in which $V$ also depends on velocities, more precisely the case $V = V_0(\mathbf{q}) + V_1(\mathbf{q}, \dot{\mathbf{q}})$, $V_1$ being linear in $\dot{\mathbf{q}}$ (Lorentz force, Coriolis force); quite clearly, we get

$$E = 2K - V_1 - (K - V_0 - V_1) = K + V_0 \ .$$

We see the quantity $E$ mantains the meaning of energy, however the term $V_1$, linear in $\dot{\mathbf{q}}$, does not contribute. This is coherent with the well known fact that the forces associated to $V_1$ are orthogonal to the velocity, and thus do not make work. They are called *gyroscopic forces*.

The Lagrangian formalism is also interesting if, in addition to conservative and gyroscopic forces, with overall potential $V(\mathbf{q}, \dot{\mathbf{q}})$, there exist other forces, of any nature, with Lagrangian components $Q_1, \ldots, Q_n$. In such a case, as we already commented, the Lagrange equations can be written in the mixed form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = Q_h(\mathbf{q}, \dot{\mathbf{q}}, t) \ ,$$

with $L = K - V$. Proceeding as above, we obtain (for time-independent $L$), in place of the conservation law $\dot{E} = 0$, the more general relation

$$\dot{E} = \sum_{h=1}^{n} Q_h(\mathbf{q}, \dot{\mathbf{q}}, t)\, \dot{q}_h \ ;$$

the right hand side represents the power of the additional forces, written in the free coordinates. If such a quantity is always less or equal zero (as in the case of friction), the forces are called *dissipative.*

Of course, we can also include in $Q_h$ all active forces, thus making reference to the Lagrange equations in their improper form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} = Q_h \ ;$$

in such a case, by applying the previous result with $V = 0$ and $E = K$, we obtain

$$\dot{K} = \sum_{h=1}^{n} \dot{q}_h Q_h(\mathbf{q}, \dot{\mathbf{q}}, t) \ ;$$

this is the well know theorem of kinetic energy (also called work–energy theorem), stating that the time derivative of the kinetic energy is equal to the power of all active forces.

### 2.6.2 Ignorable coordinates and reduction

Consider a Lagrangian system with $n$ degrees of freedom, and assume $L$ does not depend on certain coordinates,[18] for example $q_{m+1}, \dots, q_n$, for some $m < n$. If we denote $\mathbf{q}' = (q_1, \dots, q_m)$, $\mathbf{q}'' = (q_{m+1}, \dots, q_n)$, then the Lagrangian does not depend on $\mathbf{q}''$, and can be written

$$L(\mathbf{q}', \dot{\mathbf{q}}', \dot{\mathbf{q}}'', t) \ .$$

Such a situation arises, typically, in problems that include some symmetry, if the free coordinates are chosen appropriately; for example, for a problem of central motion like the Kepler problem, if we reduce to the plane of the orbit and use the polar coordinates $r, \vartheta$, then the Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r)$ does not depend on $\vartheta$.

If the Lagrangian does not depend on a given coordinate $q_l$, its conjugate momentum $p_l = \frac{\partial L}{\partial \dot{q}_l}$, see (2.4.9), according to Lagrange equations, is conserved; in the example, the conserved quantity is $p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta}$, that is the angular momentum. If, as we assumed above, the Lagrangian does not depend on $q_{m+1}, \dots, q_n$, then there are $n - m$ constants of motion

$$p''_l(\mathbf{q}', \dot{\mathbf{q}}', \dot{\mathbf{q}}'', t) = \frac{\partial L}{\partial \dot{q}_l}(\mathbf{q}', \dot{\mathbf{q}}', \dot{\mathbf{q}}'', t) \ , \qquad l = m+1, \dots, n \ . \tag{2.6.4}$$

We wish to exploit the existence of such constants of motion to reduce the number of the effective degrees of freedom of the system from $n$ to $m$, that is to write a Lagrangian $L'$ which depends only on $\mathbf{q}'$, $\dot{\mathbf{q}}'$, but concerning the first $m$ coordinates is equivalent to $L$; $L'$ will of course depend also on the momenta $p''_l$, with the role of constant parameters. To let $\dot{\mathbf{q}}''$ disappear, and let instead the vector $\mathbf{p}'' = (p_{m+1}, \dots, p_n)$ appear, it is necessary to invert equations (2.6.4) with respect to $\dot{q}''_{m+1}, \dots, \dot{q}''_n$, so as to obtain an expression of the form

$$\dot{\mathbf{q}}'' = \mathbf{u}(\mathbf{q}', \dot{\mathbf{q}}', t, \mathbf{p}'') \ .$$

The inversion is always possible, and explicit, for natural systems,[19] that is in the familiar case $L = K - V$ with $K = K_2 + K_1 + K_0$. Indeed in such a case it is easy to see that (2.6.4) is a linear relation of the form

$$\mathbf{p}'' = a''(\mathbf{q}', t)\dot{\mathbf{q}}'' + \mathbf{f}(\mathbf{q}', \dot{\mathbf{q}}', t) \ , \tag{2.6.5}$$

where $a''$ is the diagonal minor of the kinetic matrix formed by the last $n - m$ rows and columns, while $\mathbf{f}$ is a convenient function (not containing $\dot{\mathbf{q}}''$). *Thanks to the fact that the kinetic matrix is positive definite,* we have, in particular, $\det a'' \neq 0$, and thus (2.6.5) can be inverted, giving

$$\dot{\mathbf{q}}'' = a''(\mathbf{q}', t)^{-1}[\mathbf{p}'' - \mathbf{f}(\mathbf{q}', \dot{\mathbf{q}}', t)] \ . \tag{2.6.6}$$

The right hand side of (2.6.6) is the function $\mathbf{u}$ we were looking for.

[18] At least for natural systems, the Lagrangian must instead depend on all velocities, otherwise the kinetic matrix is not positive definite.

[19] In the case of general Lagrangian systems, mathematical analysys tells us that the condition which in principle assures, at least locally, the invertibility, is that the jacobian matrix

$$\Big(\frac{\partial p_l}{\partial \dot{q}_k}\Big)_{l,k=m+1,\dots n} = \Big(\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_k}\Big)_{l,k=m+1,\dots n}$$

has nonvanishing determinant. "In principle" means that the function $\mathbf{u}$ is by itself well defined, not that we are able to write it explicitely. In the case of natural systems, instead, the inversion is always explicit and global.

This expression can be used to replace $\dot{\mathbf{q}}''$ in $L$, thus eliminating it. *The Lagrangian* $L(\mathbf{q}', \dot{\mathbf{q}}', \mathbf{u}(\mathbf{q}', \dot{\mathbf{q}}', t, \mathbf{p}''), t)$ *obtained in this way, however, is not yet the good one,* and a correction is needed: precisely, we should pose

$$L'(\mathbf{q}', \dot{\mathbf{q}}', t, \mathbf{p}'') = L(\mathbf{q}', \dot{\mathbf{q}}', \mathbf{u}(\mathbf{q}', \dot{\mathbf{q}}', t, \mathbf{p}''), t) - \mathbf{p}'' \cdot \mathbf{u}(\mathbf{q}', \dot{\mathbf{q}}', t, \mathbf{p}'') \ .$$

The following important proposition, known as *Routh's theorem,* guarantees indeed this is the appropriate Lagrangian:

PROPOSITION 31 *For any solution* $\mathbf{q}(t) = (\mathbf{q}'(t), \mathbf{q}''(t))$ *of the Lagrange equations relative to* $L$, $\mathbf{q}'(t)$ *solves the Lagrange equations relative to* $L'$, *while* $\mathbf{q}''(t)$ *is given by*

$$\mathbf{q}''(t) = \mathbf{q}''(0) + \int_0^t \mathbf{u}(\mathbf{q}'(s), \dot{\mathbf{q}}'(s), s, \mathbf{p}'') \, \mathrm{d}s \ . \tag{2.6.7}$$

PROOF. For $h \leq m$ we have

$$\frac{\partial L'}{\partial q_h} = \frac{\partial L}{\partial q_h} + \sum_{l=m+1}^{n} \frac{\partial L}{\partial \dot{q}_l} \frac{\partial u_l}{\partial q_h} \ - \sum_{l=m+1}^{n} p_l \frac{\partial u_l}{\partial q_h} = \frac{\partial L}{\partial q_h} \ ,$$

and similarly $\frac{\partial L'}{\partial \dot{q}_h} = \frac{\partial L}{\partial \dot{q}_h}$; as a consequence, the Lagrange equations relative to $L'$ coincide with the first $m$ Lagrange equations relative to $L$. Equation (2.6.7) is obvious. □

The system with only $m$ degrees of freedom described by $L'$, is called the *reduced system;* $L'$ in turn is called the *reduced Lagrangian*, and the overall procedure we followed is called *reduction*. The coordinates $q_{m+1}, \ldots, q_n$ are called *ignorable coordinates.* The proposition shows that the solution of the reduced problem is equivalent to the resolution of the complete problem. Properly speaking, $L'$ is a family of Lagrangians parametrized by the $n-m$ parameters $p_{m+1}, \ldots, p_n$ (which are determined by the initial datum).

EXAMPLE In the above considered example of the central motion ($n = 2$, $m = 1$), it is, as already commented, $p_\vartheta = mr^2\dot{\vartheta}$ (the relation is linear homogeneous, and correspondingly $p_\vartheta$ does not depend on $\dot{r}$). The inversion trivially gives

$$\dot{\vartheta} = u(r, p_\vartheta) = \frac{p_\vartheta}{mr^2}$$

($u$ in this case does not contain $\dot{r}$), and finally we have

$$\begin{aligned} L'(r, \dot{r}, p_\vartheta) &= \frac{1}{2} m\dot{r}^2 + \frac{p_\vartheta^2}{2mr^2} - V(r) - \frac{p_\vartheta^2}{mr^2} \\ &= \frac{1}{2} m\dot{r}^2 - W(r; p_\vartheta) \ , \end{aligned}$$

with

$$W(r, p_\vartheta) = V(r) + \frac{p_\vartheta^2}{2mr^2} \ .$$

We thus obtained a Lagrangian system with only one degree of freedom, with an "effective potential energy" $W$; observe the additional term (for $p_\vartheta \neq 0$) is repulsive and diverges in the origin. If $V$ is the Kepler potential $V(r) = -k/r$, the graphic of the effective potential $W$ and the phase portrait of

the reduced system are (apart for notations) as in figure 1.11. Yet we did not discuss systematically, in the Lagrangian context, about equilibrium, but the phase portrait clearly shows that for any $p_\vartheta \neq 0$ the reduced system has has a unique equilibrium point $r^*$; correspondingly, the complete system has a uniform circular motion of radius $r^*$, with

$$\dot{\vartheta} = \frac{p_\vartheta}{m(r^*)^2} \ .$$

Expressions (2.6.6) and (2.6.7) show that always, when the reduced system has an equilibrium (constant $\mathbf{q}'$ and $\dot{\mathbf{q}}' = 0$), the ignorable coordinates advance uniformly.

EXERCISE 35 *Draw the graphic of the effective potential $W$ for the two-dimensional harmonic oscillator and for a point mass subject to a radial force of constant intensity (point constrained to the surface of a reversed cone, in the gravity). Make a qualitatively study of the motion in the two cases.*

EXERCISE 36 *Determine the conditions on $V(r)$, such that the origin cannot be reached (for fixed energy and angular momentum).*

### 2.6.3 The Noether's theorem

Consider a Lagrangian system of Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$; let

$$\mathbf{q} \mapsto \boldsymbol{\varphi}(\alpha, \mathbf{q}) \tag{2.6.8}$$

be a transformation dependent on a parameter $\alpha$, defined for $\alpha$ in a neighborhood of the origin, which for $\alpha = 0$ is the identity:

$$\boldsymbol{\varphi}(0, \mathbf{q}) = \mathbf{q} \ .$$

For any fixed $\mathbf{q}$, by varying $\alpha$, the map (2.6.8) defines an arc of a curve passing through $\mathbf{q}$ in the configurations space (figure 2.6, left). The map naturally extends to the velocities by posing

$$\dot{\mathbf{q}} \mapsto \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}) \ , \tag{2.6.9}$$

where

$$\boldsymbol{\psi} = \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} \qquad \text{i.e.} \qquad \psi_h(\alpha, \mathbf{q}, \dot{\mathbf{q}}) = \sum_{k=1}^{n} \frac{\partial \varphi_h}{\partial q_k}(\alpha, \mathbf{q}) \ \dot{q}_k \ . \tag{2.6.10}$$

For $\alpha = 0$, $\boldsymbol{\psi}$ too is the identity (the Jacobian matrix $\frac{\partial \varphi_h}{\partial q_k}$, which appears in (2.6.10), for $\alpha = 0$ is the identity matrix); so, by varying $\alpha$, the extended map $(\boldsymbol{\varphi}(\alpha, \mathbf{q}), \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}))$ defines a curve in the $2n$ dimensional space of states, which for $\alpha = 0$ passes through $(\mathbf{q}, \dot{\mathbf{q}})$ (figure 2.6, right).

There are relevant examples, connected with symmetries of the system at hand, in which the Lagrangian remains invariant, that is stays constant, along similar curves:

$$L(\boldsymbol{\varphi}(\alpha, \mathbf{q}), \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}), t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) \ . \tag{2.6.11}$$

An elementary example is the case discussed above, in section 2.6.2, where the Lagrangian does not depend on a particular coordinate $q_l$: quite evidently, the absence of $q_l$ among the arguments of $L$, is equivalent to say that $L$ is invariant for the particular transformation

$$\varphi_h(\alpha, \mathbf{q}) = q_h + \alpha \delta_{lh} \ , \qquad \psi_h(\alpha, \mathbf{q}, \dot{\mathbf{q}}) = \dot{q}_h \ , \qquad h = 1, \ldots, n \ , \tag{2.6.12}$$

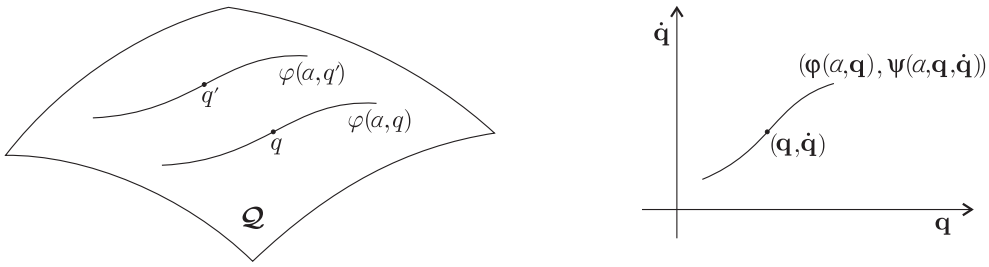


Figure 2.6: *Illustrating the Noether's theorem.*

in which the (only) coordinate $q_l$ is translated by $\alpha$. To this invariance property, we have seen, the law of conservation of the momentum $p_l$ is associated.

Such a result generalizes to any transformation of the form (2.6.8), which leaves $L$ invariant: more precisely, *to each invariance property, a particular conservation law is associated.* This is ensured by the following proposition, which is a particular case of a more general theorem due to Emmy Noether:

PROPOSITION 32 *Consider a family of regular maps $\mathbf{q} \mapsto \boldsymbol{\varphi}(\alpha, \mathbf{q})$, depending on a real parameter $\alpha$, defined and regular in $\alpha$ for $\alpha$ in a neighborhood of the origin, such that $\boldsymbol{\varphi}(0, \mathbf{q}) = \mathbf{q}$; let (2.6.9) be its natural extension to the velocities. If for any choice of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\alpha$ it is*

$$L(\boldsymbol{\varphi}(\alpha, \mathbf{q}), \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}), t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) \ , \tag{2.6.13}$$

*then the function*

$$P(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{h=1}^{n} \frac{\partial \varphi_h}{\partial \alpha}(0, \mathbf{q})\, p_h(\mathbf{q}, \dot{\mathbf{q}}, t) \ ,$$

*where $p_h = \frac{\partial L}{\partial \dot{q}_h}$, is a constant of motion for the Lagrange equations associated to $L$.*

For the translation (2.6.12) it is $P = p_l$; in general instead $P$ is a linear combination of momenta, with coefficients $\frac{\partial \varphi_h}{\partial \alpha}$ (to be evaluated at $\alpha = 0$).

PROOF. According to (2.6.13), the derivative of $L(\boldsymbol{\varphi}(\alpha, \mathbf{q}), \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}), t)$ with respect to $\alpha$ vanishes:

$$\sum_{h=1}^{n} \Big[ \frac{\partial L}{\partial q_h}(\boldsymbol{\varphi}(\alpha, \mathbf{q}), \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}), t)\, \frac{\partial \varphi_h}{\partial \alpha}(\alpha, \mathbf{q}) + \frac{\partial L}{\partial \dot{q}_h}(\boldsymbol{\varphi}(\alpha, \mathbf{q}), \boldsymbol{\psi}(\alpha, \mathbf{q}, \dot{\mathbf{q}}), t)\, \frac{\partial \psi_h}{\partial \alpha}(\alpha, \mathbf{q}, \dot{\mathbf{q}}) \Big] = 0 \ .$$

From the definition $\psi_h = \frac{\mathrm{d}\varphi_h}{\mathrm{d}t}$ it follows (lemma 22) $\frac{\partial \psi_h}{\partial \alpha} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \varphi_h}{\partial \alpha}$; by substituting, and posing $\alpha = 0$, we then obtain

$$\sum_{h=1}^{n} \Big[ \frac{\partial L}{\partial q_h}(\mathbf{q}, \dot{\mathbf{q}}, t) \frac{\partial \varphi_h}{\partial \alpha}(0, \mathbf{q}) + \frac{\partial L}{\partial \dot{q}_h}(\mathbf{q}, \dot{\mathbf{q}}, t) \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \varphi_h}{\partial \alpha}(0, \mathbf{q}) \Big] = 0 \ .$$

Recalling finally that, along any solution of the Lagrange equations, it is $\frac{\partial L}{\partial q_h} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_h}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{h=1}^{n} \frac{\partial \varphi_h}{\partial \alpha}(0, \mathbf{q})\, \frac{\partial L}{\partial \dot{q}_h}(\mathbf{q}, \dot{\mathbf{q}}, t) = 0 \ ,$$

that is $\dot{P} = 0$. □

The basic example to understand Noether's theorem is the following:

EXAMPLE Consider two (unconstrained) point masses $m_1$ and $m_2$, and suppose the interaction potential is central, namely it depends only on the distance $|P_2 - P_1|$. Denoting by $(q_1, \ldots, q_6)$ the Cartesian coordinates of $P_1$ and $P_2$, the Lagrangian is of the form

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m_1 (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \frac{1}{2} m_2 (\dot{q}_4^2 + \dot{q}_5^2 + \dot{q}_6^2) - V\big((q_4 - q_1)^2 + (q_5 - q_2)^2 + (q_6 - q_3)^2\big) \ .$$

Using Noether's system, we can show the overall momentum and angular momentum of the system are conserved.

a) $L$ is invariant under translation along any of the Cartesian axes. With reference to translations along the $x$ axis, this means $L$ stays constant under the substitution $q_h \mapsto \varphi_h(\alpha, \mathbf{q})$, $\dot{q}_h \mapsto \psi_h(\alpha, \mathbf{q}, \dot{\mathbf{q}})$, with

$$\varphi_1 = q_1 + \alpha \ , \qquad \varphi_4 = q_4 + \alpha \ , \qquad \varphi_h = q_h \quad \text{for} \quad h \neq 1, 4 \ ,$$

and correspondingly $\psi_h = \dot{q}_h$ for any $h$. Using Noether's theorem we immediately conclude that $P = p_1 + p_4$, namely the $x$ component of the linear momentum, is conserved. In the same way we can proceed for the two other components.

b) The Lagrangian, thanks to the spherical symmetry, is invariant under rotation along any of the coordinate axes. With reference to rotations along the $z$ axis, this means the substitution $q_h \mapsto \varphi_h(\alpha, \mathbf{q})$ defined by

$$\begin{aligned} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \ ; \qquad \varphi_3 = q_3 \\ \begin{pmatrix} \varphi_4 \\ \varphi_5 \end{pmatrix} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_4 \\ q_5 \end{pmatrix} \ ; \qquad \varphi_6 = q_6 \end{aligned}$$

with obvious extension $\dot{q}_h \mapsto \psi_h(\alpha, \mathbf{q}, \dot{\mathbf{q}})$, leaves $L$ invariant. In such a case we have

$$\frac{\partial \varphi_1}{\partial \alpha}(0, \mathbf{q}) = -q_2 \ , \qquad \frac{\partial \varphi_2}{\partial \alpha}(0, \mathbf{q}) = q_1 \ , \qquad \frac{\partial \varphi_3}{\partial \alpha}(0, \mathbf{q}) = 0 \ ,$$

and similar relations with all indices raised by three. The conserved quantity is then

$$P = -q_2 p_1 + q_1 p_2 - q_5 p_4 + q_4 p_5 = m_1 (x_1 \dot{y}_1 - y_1 \dot{x}_1) + m_2 (x_2 \dot{y}_2 - y_2 \dot{x}_2) \ ,$$

namely the $z$ component of the angular momentum. In the same way we can proceed for the two other components.

The example immediately generalizes to any number of point masses, interacting through internal forces of a central type. We then see that *within the Lagrangian formalism, the most common conservation laws of Physics can be traced back to the invariance properties of the Lagrangian under translation* (homogeneity of space) *and under rotation* (isotropy of space). It is worthwhile to observe that *the conservation of energy,* which requires the independence of the Lagrangian from $t$, *is also associated to an invariance property, namely the invariance of the Lagrangian for time translations* (homogeneity of time).

## 2.7 Equilibrium, stability, small oscillations

In this section we will study equilibrium, stabilty of equilibrium, and motions close to equilibrium, as they are treated within the Lagrangian formalism. Attention will be restricted to natural systems, moreover we will suppose constraints are fixed and coordinates are independent of $t$. In these assumptions, as we know, kinetic energy $K$ reduces to its quadratic part $K_2$:

$$K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{h,k=1}^{n} a_{hk}(\mathbf{q}) \dot{q}_h \dot{q}_k \ .$$

The forces, and so $V$ in the conservative case, will also be assumed to be independent of $t$.

On several occasions, we shall invoke notions and results of Chapter 1, concerning generic differential equations. To avoid confusion, it will always be necessary to keep in mind the distinction, in the Lagrangian context, between space of configurations, $n$-dimensional and endowed with coordinates $\mathbf{q}$, and space of states, $2n$-dimensional and endowed with coordinates $(\mathbf{q}, \dot{\mathbf{q}})$.

### 2.7.1 Equilibrium

Let us consider a holonomic system with $n$ degrees of freedom, with fixed constraints and forces independent of $t$, defined for $\mathbf{q}$ in some open set $\mathcal{U}_0 \subset \mathbb{R}^n$ and $\dot{\mathbf{q}} \in \mathbb{R}^n$; the kinetic energy correspondingly reduces to its quadratic part. As we know, the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} = Q_h \ , \qquad h = 1, \dots, n$$

can always be put in normal form, more precisely we can write them as a system of $n$ second order differential equations of the form $\ddot{q}_h = f_h(\mathbf{q}, \dot{\mathbf{q}})$, $h = 1, \dots, n$, or equivalently a system of $2n$ first order equations

$$\dot{q}_h = v_h \ , \qquad \dot{v}_h = f_h(\mathbf{q}, \mathbf{v}) \ , \qquad h = 1, \dots, n \ . \tag{2.7.1}$$

Taking up the general notion of equilibrium point for a system of differential equations, as introduced in Chapter 1,[20] we shall say that $\mathbf{q}^* \in \mathcal{U}_0$ is an *equilibrium configuration* for the Lagrange equations, if $\mathbf{c} = (\mathbf{q}^*, 0)$ is an equilibrium point for the system (2.7.1), that is if

$$f_h(\mathbf{q}^*, 0) = 0 \ , \qquad h = 1, \dots, n \ .$$

It is then easy to verify the following

Proposition 33 *The configuration $\mathbf{q}^* \in \mathcal{U}_0$ is an equilibrium configuration, if and only if $Q_h(\mathbf{q}^*, 0) = 0$ for $h = 1, \dots, n$.*

Proof. Let us recall (section 2.4.4, proposition 24) that $\mathbf{f} = a^{-1}(\mathbf{Q} - \mathbf{g})$, where $a$ is the kinetic matrix. In the case we are here considering, $\mathbf{g}$ vanishes for $\dot{\mathbf{q}} = 0$, and thus

$$\mathbf{f}(\mathbf{q}^*, 0) = a^{-1}(\mathbf{q}^*)\, \mathbf{Q}(\mathbf{q}^*, 0) \ .$$

As a consequence, $\mathbf{f}(\mathbf{q}^*, 0)$ vanishes if and only if $\mathbf{Q}(\mathbf{q}^*, 0)$ vanishes. □

[20] Here it is particularly important to distinguish between configuration space (to which $\mathbf{q}^*$ belongs) and space of states (to which $\mathbf{c}$ belongs); pay attention to the use we are making of the terms "configuration" and "point".

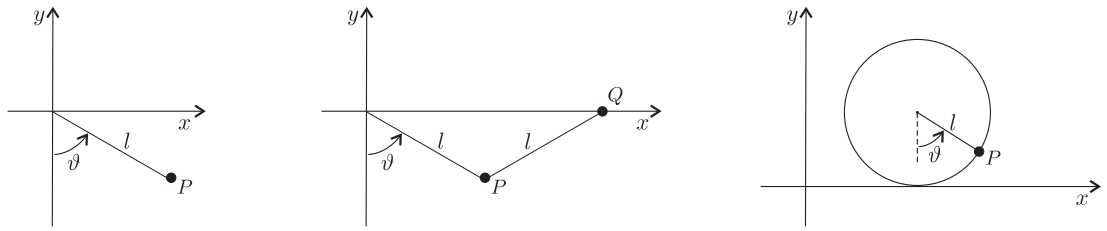


Figure 2.7: *Three systems having different kinetic energy but the same potential energy, and thus the same equilibrium configurations.*

- It clearly appears that the possible presence of forces proportional to the velocity, or more generally forces which vanish for vanishing velocities (viscous friction, Coriolis or Lorentz force), is completely irrelevant to determine the equilibrium configurations.

- Such a result is clearly the analog of the result, obvious within Newtonian mechanics, that there is equilibrium if and only if forces vanish. The proposition however is not trivial: think of a point mass constrained to a surface; in the equation of motion $m\mathbf{a} = \mathbf{F} + \mathbf{\Phi}$, the constraint reaction $\mathbf{\Phi}$ is also present, but $\mathbf{\Phi}$ does not enter the equilibrium condition, according to which it is necessary and sufficient that the components of $\mathbf{F}$ tangent to the surface do vanish.

In the particularly relevant case of conservative forces described by a potential energy $V$, it is $Q_h(\mathbf{q}) = -\frac{\partial V}{\partial q_h}(\mathbf{q})$. The above proposition assumes then the following form:

PROPOSITION 34 *In the case of conservative positional forces, the configuration $\mathbf{q}^* \in \mathcal{U}_0$ is an equilibrium configuration, if and only if $V$ is stationary in $\mathbf{q}^*$, that is $\frac{\partial V}{\partial q_h}(\mathbf{q}^*) = 0$ for any $h$.*

- So, systems having different kinetic energy, but the same potential energy, have the same equilibrium configurations. Look at the three systems represented in figure 2.7: the kinetic structure is different, the equations of motion are different, but for all of them the potential energy is the elementary one of the simple pendulum, and so, although moving differently, they share the equilibrium configurations $\vartheta = 0, \pi$.

- In this section, we are restricting the attention, for simplicity, to the case in which the kinetic energy only contains the term $K_2$. In the general mechanical case $K = K_2 + K_1 + K_0$, it is easy to see that: (i) $K_1$ is irrelevant, precisely as a potential term $V_1$ would be; (ii) proposition 34 remains valid, if $V$ is replaced by $V - K_0$.

### 2.7.2 Stability of equilibrium

Let us now come to the question of stability of equilibrium. Making reference to the notion of stability introduced in Chapter 1, we shall say that the equilibrium configuration $\mathbf{q}^* \in \mathcal{U}_0$ is stable for the Lagrange equations, if $\mathbf{c} = (\mathbf{q}^*, 0)$ is a stable equilibrium point for the system (2.7.1); in a similar obvious way, the notion of stability in the future, or in the past, as well as the notion of asymptotic stability, are also transposed.

- It is not difficult to recognize that the notion of stability can be rephrased in the following way: for any neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of $\mathbf{q}^*$, and any $\varepsilon > 0$, there exist a neighborhood $\mathcal{V}$ of $\mathbf{q}^*$ and a number $\delta > 0$, such that any motion with initial datum $(\mathbf{q}^0, \dot{\mathbf{q}}^0)$, with $\mathbf{q}^0 \in \mathcal{V}$ and $K(\mathbf{q}^0, \dot{\mathbf{q}}^0) < \delta$, stays forever in $\mathcal{U}$ and mantains kinetic energy $K(\mathbf{q}, \dot{\mathbf{q}}) < \varepsilon$. The proof is left as an exercise; it is actually enough to see that it is not restricive to take the neighborhoods $U$ and $V$, which appear in the definition of stability, of the type $U = \big\{(\mathbf{q}, \dot{\mathbf{q}}) : \mathbf{q} \in \mathcal{U}, K(\mathbf{q}, \dot{\mathbf{q}}) < \varepsilon\big\}$, and similarly for $V$).

In the case of conservative positional forces, a stability criterion of fundamental importance, in the form of a *sufficient condition* for the stability of equilibrium, is stated in the following proposition, known as the *Lagrange–Dirichlet theorem:*

PROPOSITION 35 *Consider a natural Lagrangian system, with Lagrangian* $L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$, $K = K_2 = \frac{1}{2}\sum_{h,k} a_{hk}(\mathbf{q})\dot{q}_h\dot{q}_k$. *If the potential energy* $V$ *has a proper minimum in* $\mathbf{q}^*$, *then* $\mathbf{q}^*$ *is a stable equilibrium configuration (for all times).*

PROOF. If $\mathbf{q}^*$ is a minimum for $V$, then it is $\frac{\partial V}{\partial q_h} = 0$, $h = 1, \dots, n$, and so $\mathbf{q}^*$ is an equilibrium configuration. The stability of $\mathbf{q}^*$ follows as an easy corollary of the Lyapunov theorem,[21] by simply using the energy as the Lyapunov function. To apply the theorem we must obviously refer to the $2n$–dimensional space where the motion takes place, with the correspondence of notation $\mathbf{x} = (\mathbf{q}, \dot{\mathbf{q}})$, and check that in a neighborhood of the equilibrium point $\mathbf{c} = (\mathbf{q}^*, 0)$, the energy $E = K + V$ is a good Lyapunov function. This is immediate: since $K$, as a function of $\dot{\mathbf{q}}$, is positive definite, if $V$ has a proper minimum in $\mathbf{q}^*$, then $E$ has a minimum in $\mathbf{c}$ (going out of $\mathbf{c}$, either $K$ or $V$ or both grow).[22] On the other hand, since $E$ is a constant of motion, it is $\dot{E} = 0$. This concludes the proof. □

The Lagrange–Dirichlet theorem extends in an obvious way to the most common cases of velocity dependent forces. Precisely:

i) In presence of gyroscopic forces, that is for velocity dependent potentials of the form $V(\mathbf{q}, \dot{\mathbf{q}}) = V_0(\mathbf{q}) + V_1(\mathbf{q}, \dot{\mathbf{q}})$, with $V_1$ linear in $\dot{\mathbf{q}}$, it is easy to see that the stability criterion remains valid, as a sufficient condition, provided we make reference only to the positional part $V_0$ of the potential: indeed, in a neighborhood of a minimum of $V_0$ (which remains an equilibrium configuration), the energy $E = K + V_0$ (recall the term $V_1$ does not contribute to $E$) remains a good Lyapunov function, as if velocity dependent forces did not exist. So, in particular, the Lagrange-Dirichlet stability condition extends to the case of magnetic forces and of rotating reference frames. *Warning:* we are not saying gyroscopic forces are irrelevant for the stability of equilibrium, but only that if $V_0$ has a proper minuimum in $\mathbf{q}^*$, then adding a gyroscopic force cannot make $\mathbf{q}^*$ unstable. Conversely, however, *gyroscopic force can produce stability:* there are indeed cases (see appendix C) in which $\mathbf{q}^*$ is not stable, but gets stable thanks to the introduction of a convenient gyroscopic force.

[21] Hystorically, things went in the opposite way, that is the Lyapunov theorem came as an extension of the Lagrange–Dirichlet theorem.

[22] Observe that, should instead $V$ have a maximum, $E$ would be neither maximum nor minimum in $\mathbf{c}$, but a saddle-point.

ii) If, in addition to conservative and possibly gyroscopic forces, there are dissipative forces, the stability of a proper minimum of $V_0$ persists, for positive times. Indeed, as we have seen, dissipative forces are characterized by the fact that $\dot{E} = \sum_h \dot{q}_h Q_h \leq 0$; for the Lyapunov theorem, the conclusion is immediate. In typical cases of physical interest (in particular for viscous friction), using as hypothesis inside the Lyapunov theorem the assumption b"), weaker than b'), it is possible to show that dissipation not only does not destroy stability (in the future), but converts it in asymptotic stability.

It is worthwhile to stress that the above proved Lagrange–Dirichlet theorem gives only a sufficient condition, and not a necessary one,[23] for the stability of equilibrium. There is however an important case in which the condition is also necessary: namely the case in which gyroscopic forces are not present, and (as is generic) the presence or absence of a minmum of $V$ in $\mathbf{q}^*$ is decided by looking only at the second derivatives of $V$, without necessity to investigate the higher order derivatives.

Let us denote by $B$ the Hessian matrix of $V$, computed in the equilibrium configuration $\mathbf{q}^*$:

$$B_{hk} = \frac{\partial^2 V}{\partial q_h \partial q_k}(\mathbf{q}^*) \ .$$

If $B$ is positive definite, then $V$ has a minimum in $\mathbf{q}^*$, and correspondingly all eigenvalues of $V$ are positive. If instead $B$ has one or more negative eigenvalues, then certainly $V$ does not have a minimum in $\mathbf{q}^*$ (going out of $\mathbf{q}^*$ in the direction corresponding to a negative eigenvalue, $V$ decreases); correspondingly, as we shall see, it is possible to prove that $\mathbf{q}^*$ is not stable. The only case that escapes such analysis, in which only the Hessian matrix i.e. the second derivatives of $V$ are considered, is the case in which all eigenvalues of $B$ are non-negative, and at least one is zero.

Other necessary conditions for the stability of an equilibrium are known, but the general problem of finding a necessary condition for the stability of an equilibrium configuration (known as the *inverse Dirichet problem*) is still partially open.

- For $n = 2$, deciding if a symmetric matrix $B$ is positive definite, is immediate: it is enough to write the eigenvalue equation, and pretend both eigenvalues are positive. It is also immediate to recognize that:
  - If $\det B > 0$, then the matrix is definite (either positive or negative). Indeed in such a case the eigenvalues are both positive or both negative: positive if the diagonal elements of $B$ (that for $\det B > 0$ have the same sign) are positive, negative if they are negative; it is enough to look at any of them (or at their sum, i.e. the trace of $B$, as is occasionally easier).
  - if instead $\det B \leq 0$, the matrix is not definite.

### 2.7.3 Linearization of the equations around an equilibrium point

Our aim here is to study the behavior of Lagrangian systems near an equilibrium point. For this purpose, we shall apply to the Lagrange equations a linearization procedure, similar to the one we used in the first chapter to pass from the equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ to the linearized equation $\dot{\mathbf{x}} = A\mathbf{x}$,

[23] A simple counterexample is provided by a system with only one degree of freedom, with potential energy $V(q) = q^k \sin q^{-1}$, $k > 4$ (completed by $V(0) = 0$): the origin is not a minimum of $V$, nevertheless it is easy to recognize it is stable. A counterexample with $V$ of class $C^\infty$ is given by $V(q) = \exp(-1/q^2) \sin q^{-1}$.

in view of the classification of critical points. We should however keep in mind a few differences: indeed, unlike Chapter 1, here we will deal directly with second order equations; moreover, we will restrict the attention to conservative systems. In the essence, what we will do is extending to Lagrangian systems with $n$ degrees of freedom the study we made in Chapter 1 for conservative systems with one degree of freedom (systems $\dot{\mathbf{x}} = A\mathbf{x}$ in $\mathbb{R}^2$, with $\operatorname{Tr} A = 0$).

So, let us consider a natural Lagrangian system with $n$ degrees of freedom, with time independent Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$, $K = K_2 = \frac{1}{2}\sum_{h,k} a_{hk}(\mathbf{q})\dot{q}_h\dot{q}_k$. We assume the system has an equilibrium configuration $\mathbf{q}^*$, and expand $L$ around $\mathbf{c} = (\mathbf{q}^*, 0)$, considering small both the difference $\mathbf{q} - \mathbf{q}^*$ and the velocity $\dot{\mathbf{q}}$. To simplify notations, without loss of generality (it is enough to translate the origin), we shall assume $\mathbf{q}^* = 0$. We then find

$$\begin{aligned} a_{hk}(\mathbf{q}) &= a_{hk}(0) + \mathcal{O}(\|\mathbf{q}\|) \\ K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\sum_{h,k=1}^{n} a_{hk}(0)\dot{q}_h\dot{q}_k + \mathcal{O}(\|\mathbf{q}\|\,\|\dot{\mathbf{q}}\|^2) \ , \end{aligned}$$

while concerning $V$ it is

$$V(\mathbf{q}) = V(0) + \sum_{h=1}^{n} \frac{\partial V}{\partial q_h}(0) q_h + \frac{1}{2}\sum_{h,k=1}^{n} \frac{\partial^2 V}{\partial q_h \partial q_k}(0) q_h q_k + \mathcal{O}(\|\mathbf{q}\|^3) \ .$$

Neglecting the constant $V(0)$, and recalling $\frac{\partial V}{\partial q_h}(0) = 0$ for any $h$, we then obtain for $L$ the expansion

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\sum_{h,k=1}^{n} a_{hk}(0)\dot{q}_h\dot{q}_k - \frac{1}{2}\sum_{h,k=1}^{n} \frac{\partial^2 V}{\partial q_h \partial q_k}(0) q_h q_k + \mathcal{O}(\|(\mathbf{q}, \dot{\mathbf{q}})\|^3) \ .$$

Such a Lagrangian has the form $L = L^* + \mathcal{O}(\|(\mathbf{q}, \dot{\mathbf{q}})\|^3)$, with

$$L^*(\mathbf{q}, \dot{\mathbf{q}}) = K^*(\dot{\mathbf{q}}) - V^*(\mathbf{q}) \ , \tag{2.7.2}$$

having set

$$\begin{aligned} K^* &= \frac{1}{2}\sum_{h,k=1}^{n} A_{hk}\dot{q}_h\dot{q}_k \ , \qquad A_{hk} = a_{hk}(0) \\ V^* &= \frac{1}{2}\sum_{h,k=1}^{n} B_{hk} q_h q_k \ , \qquad B_{hk} = \frac{\partial^2 V}{\partial q_h \partial q_k}(0) \ ; \end{aligned}$$

in more compact notation we can write

$$L^*(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}} \cdot A\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q} \cdot B\mathbf{q} \ .$$

The equations of motion associated to $L^*$ are linear, precisely have the form

$$A\ddot{\mathbf{q}} + B\mathbf{q} = 0 \ . \tag{2.7.3}$$

It is easy to verify that the same equation is obtained if we first write the Lagrange equations corresponding to the complete Lagrangian $L$, and then linearize the equations around the equilibrium point (*Exercise:* check explicitly such a statement; observe how terms $\mathcal{O}(\|(\mathbf{q}, \dot{\mathbf{q}})\|^3)$ in $L$ necessarily produce terms $\mathcal{O}(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)$ in the equations, which disappear in the linearization).

- The linearized equations of motion (2.7.3) differ from the true equations by terms of second order in $q_h$ and $\dot{q}_h$; therefore, we chan think (but it is a delicate question, not to be taken lightly) they represent well the system in a small neighborhood of the equilibrium point. This is particularly interesting if the equilibrium point is stable, because then, as we know, any trajectory with initial datum sufficiently close to the equilibrium point stays indefinitely close to it, and so the linearized equations approximate the true equations for long times (here however we need ever greater caution: in particular, we should not think that the solution of the linearized equations and of the true equations stay necessarily close for long times).

The equations of motion (2.7.3) are linear homogeneous, therefore the superposition principle holds, and to find the general solution of the equations it is enough to find $2n$ independent particular solutions. With a typical procedure, similar to the one we used in Chapter 1 (section 1.4.2), we search for solutions of the particular factorized (or "separated") form

$$\mathbf{q}(t) = \tau(t)\mathbf{u} \ ,$$

where $\mathbf{u} \in \mathbb{R}^n$ is a constant vector, while the scalar function $\tau : \mathbb{R} \to \mathbb{R}$ encompasses the dependence on time. By substituting in (2.7.3) we obtain $\ddot{\tau}(t)A\mathbf{u} + \tau B\mathbf{u} = 0$, and this is possible if and only if $A\mathbf{u}$ and $B\mathbf{u}$ are parallel, that is if, with suitable constant $\lambda$, it is[24]

$$B\mathbf{u} = \lambda A\mathbf{u} \ . \tag{2.7.4}$$

For any value of $\lambda$ which solves this equation, the function $\tau(t)$ is then determined by the familiar second order equation

$$\ddot{\tau} = -\lambda\tau \tag{2.7.5}$$

which, we know, always furnishes two independent solutions. So, for any solution of (2.7.4), we find two independent solutions of (2.7.3); this means the search of the general solution of (2.7.3) reduces to the search of $n$ independent solutions of (2.7.4). Such an equation appears to be a generalization of the familiar eigenvalue equation for matrix $B$, to which it reduces in the case $A$ is the identity. *Thanks to the fact that $A$ is symmetric and positive definite, the generalized equation mantains, with minor adjustments, the essential algebric properties of the eigenvalue equation.* In particular (using, with little abuse, the terms eigenvalue and eigenvector respectively for $\lambda$ and $\mathbf{u}$) the following properties hold:

i) The eigenvalues are the roots of the *secular equation*

$$\det(B - \lambda A) \ = \ 0 \ , \tag{2.7.6}$$

which is an algebric equation of degree $n$ for $\lambda$.

ii) If $B$, as in our case, is symmetric, then the eigenvalues $\lambda_1, \ldots, \lambda_n$ are real, and the corresponding eigenvectors $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}$, that for real $\lambda$ can be assumed to be real, can be taken "orthonormal with respect to matrix $A$", in the sense they satisfy the condition

$$\mathbf{u}^{(i)} \cdot A\mathbf{u}^{(j)} = \delta_{ij} \ . \tag{2.7.7}$$

[24] We could also write $A\mathbf{u} = \lambda B\mathbf{u}$, excluding however the case, which is fairly possible, $B\mathbf{u} = 0$ with $A\mathbf{u} \neq 0$. The opposite case $A\mathbf{u} = 0$ with $B\mathbf{u} \neq 0$, excluded by (2.7.4), instead cannot arise ($A\mathbf{u}$ never vanishes for $\mathbf{u} \neq 0$).

iii) If $B$ too is positive definite, then the eigenvalues $\lambda_1, \ldots, \lambda_n$ are positive.

iv) If we denote by $U$ the matrix having as columns the eigenvectors, that is the matrix with entries $U_{ij} = u_i^{(j)}$, then matrix $U^T A U$ coincides with the identity, and simultaneously matrix $U^T B U$ is diagonal, precisely $(U^T B U)_{ij} = \lambda_i \delta_{ij}$.

All these properties are proved as in the case of the ordinary eigenvalue equation, with minor changes. Shortly:

– Property i) follows from the fact the homogeneous system $(B - \lambda A)\mathbf{u} = 0$ must have nontrivial solutions.

– Property ii), for the part concerning the reality of eigenvalues and eigenvectors, is obtained by writing, besides (2.7.4), the complex conjugated equation

$$B\overline{\mathbf{u}} = \overline{\lambda} A \overline{\mathbf{u}} \ ;$$

if we scalar multiply (2.7.4) by $\overline{\mathbf{u}}$ and the conjugated equation by $\mathbf{u}$, and subtract, thanks to the symmetry of the matrices we get

$$0 = (\lambda - \overline{\lambda})\, \overline{\mathbf{u}} \cdot A\mathbf{u} \ .$$

Because of the positivity of $A$,[25] it is $\overline{\mathbf{u}} \cdot A\mathbf{u} \neq 0$, and thus $\lambda - \overline{\lambda} = 0$. Concerning eigenvectors, we can certainly take them real, indeed the real and the imaginary part of complex $\mathbf{u}$, if $\lambda$ is real, are real eigenvectors with the same $\lambda$.

– Property ii), for the part concerning the othogonality (with reference to $A$) of the eigenvectors, is easy for eigenvectors $\mathbf{u}^{(i)}$, $\mathbf{u}^{(j)}$ corresponding to different eigenvalues $\lambda_i$, $\lambda_j$: indeed if

$$B\mathbf{u}^{(i)} = \lambda_i A\mathbf{u}^{(i)} \ , \qquad B\mathbf{u}^{(j)} = \lambda_j A\mathbf{u}^{(j)} \ ,$$

multiplying the former equation by $\mathbf{u}^{(j)}$ and the latter by $\mathbf{u}^{(i)}$ and subtracting, thanks to the symmetry of $A$ and $B$ we get

$$0 = (\lambda_i - \lambda_j)\, \mathbf{u}^{(j)} \cdot A\mathbf{u}^{(i)} \ ;$$

for $\lambda_i \neq \lambda_j$, orthogonality follows. The case of multiple eigenvalues is more delicate, and we shall not enter it (as for the usual eigenvalue problem, the point is showing that the geometric and algebric multiplicity do coincide; once this is established, it is easy to choose the eigenvectors, inside the eigenspace, so as the generalized othogonality is satisfied).

– Property iii) is immediate: multiplying (2.7.4) by $\mathbf{u}$ we get

$$\mathbf{u} \cdot B\mathbf{u} = \lambda\, \mathbf{u} \cdot A\mathbf{u} \ ,$$

and consequently, if $B$ is also positive definite, $\lambda$ is positive.

– Finally, concerning property iv), it is easy to see that

$$(U^T A U)_{ij} = \mathbf{u}^{(i)} \cdot A\mathbf{u}^{(j)} \ , \qquad (U^T B U)_{ij} = \lambda_j \mathbf{u}^{(i)} \cdot A\mathbf{u}^{(j)} \ ,$$

and thanks to (2.7.7), the conclusion is immediate.

[25] If the real symmetric matrix $A$ is positive definite, then by definition it is $\mathbf{u} \cdot A\mathbf{u} > 0$ for any real $\mathbf{u} \neq 0$; it immediately follows $\overline{\mathbf{u}} \cdot A\mathbf{u} > 0$ for any complex $\mathbf{u} \neq 0$ (just write $\mathbf{u} = \mathbf{v} + i\mathbf{w}$, with real $\mathbf{v}$, $\mathbf{w}$).

### 2.7.4 Normal modes of oscillation and normal coordinates

The most interesting case is when the equilibrium point is stable, more precisely $V$ has in $\mathbf{q}^* = 0$ a minimum of order two. In such a case matrix $B$ is positive definite and we can pose

$$\omega_i^2 = \lambda_i > 0 \ , \qquad i = 1, \ldots, n \ .$$

Correspondingly, for any $i$ equation (2.7.5) is the equation of a harmonic oscillator of angular frequency $\omega_i$, and the general solution of the equation can be written $\tau^{(i)}(t) = \mathcal{A}_i \cos(\omega_i t + \varphi_i)$. In turn, the general solution of (2.7.3) can be written in the form

$$\mathbf{q}(t) = \sum_{i=1}^{n} \mathcal{A}_i \cos(\omega_i t + \varphi_i)\mathbf{u}^{(i)} \ . \tag{2.7.8}$$

Observe the expression contains $2n$ arbitrary constants, whose choice is equivalent to the choice of the initial datum for $(\mathbf{q}, \dot{\mathbf{q}})$. An interesting case is when only one of the amplitudes $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is different from zero, say $\mathcal{A}_j = 1$ and $\mathcal{A}_i = 0$ for $i \neq j$: as (2.7.8) shows, in such a case we find particular solutions of the form

$$\mathbf{q}(t) = \cos(\omega_j t + \varphi_j)\mathbf{u}^{(j)} \ ,$$

with components

$$q_h(t) = U_{hj} \cos(\omega_j t + \varphi_j) \ , \qquad h = 1, \ldots, n \ ;$$

*such particular solutions are periodic,* actually harmonic, and all variables $q_1, \ldots, q_n$ oscillate with the same period and moreover with the same phase. Periodicity of motion is an exceptional fact, which disappears if two or more amplitudes are different from zero (unless the corresponding frequencies are two by two commensurable: check it as an exercise). Such particular periodic motions of the system are named *normal modes of oscillation,* and are of fundamental importance in any field of physics or engineering, wherever there are oscillating systems of any nature (from antennas to skyscrapers to musical instruments). Equation (2.7.8) shows the general solution of the linearized Lagrange equations is a superposition of normal modes.

It is also interesting to introduce the change of coordinates $\mathbf{q} = U\mathbf{x}$ in the truncated Lagrangian (2.7.2). We see immediately that the new Lagrangian $\tilde{L}(\mathbf{x}, \dot{\mathbf{x}}) = L^*(U\mathbf{x}, U\dot{\mathbf{x}})$ is

$$\tilde{L} = \frac{1}{2}(U^T A U)\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - \frac{1}{2}(U^T B U)\mathbf{x} \cdot \mathbf{x} \ .$$

But thanks to property iv) above, it is

$$\tilde{L} = \frac{1}{2}\sum_{i=1}^{n} \dot{x}_i^2 - \frac{1}{2}\sum_{i=1}^{n} \lambda_i x_i^2 \ , \tag{2.7.9}$$

and so the equations of motion decouple, namely

$$\ddot{x}_i \ = \ -\lambda_i x_i \ , \qquad i = 1, \ldots, n \ . \tag{2.7.10}$$

The coordinates $x_1, \ldots, x_n$ are named *normal coordinates* of the system. Equations (2.7.8) and (2.7.9) show a property of capital importance: *any Lagrangian system, if linearized around a stable equilibrium point* (whose stability is recognized by analyzing the second derivatives of $V$) *is*

*equivalent, via a linear change of coordinates, to a system of uncoupled harmonic oscillators.* Also observe that the energy of the system assumes the form

$$E = \sum_{i=1}^{n} E_i \ , \qquad E_i = \frac{1}{2}(\dot{x}_i^2 + \omega_i^2 x_i^2) \ ,$$

as in a system of $n$ material oscillators.

The change to normal coordinates can be done, of course, independently of the sign of the eigenvalues $\lambda_i$: in any case we obtain equations (2.7.10), decoupled from each other, which represent a harmonic oscillator ($\lambda_i > 0$), a harmonic repulsor ($\lambda_i < 0$), or a free particle ($\lambda_i = 0$). The case discussed above, in which all eigenvalues $\lambda_i$ are positive, is however the most interesting one.

- *Simultaneous diagonalization of two matrices.* As is known, the diagonalization of a single symmetric matrix can always be realized by means of an orthogonal change of coordinates (a rigid movement of the frame). Instead the simultaneous diagonalization of two matrices $A$ and $B$, unless they commute, cannot be realized by an orthogonal transformation. The simultaneous diagonalization is however possible, as we have seen, by a more general linear transformation, provided both matrices are symmetric and one at least is positive definite. To get convinced, and consider the property even obvious, few geometric considerations are sufficient. Consider, just for simplicity, the case $n = 2$, and recall that to any symmetric matrix $M$, a conic section is associated, namely the conic section of equation

  $$\mathbf{x} \cdot M\mathbf{x} = 1 \ ;$$

  the orthogonal transformation which diagonalizes $M$ is a rotation, which leads the coordinate axes to coincide with the symmetry axes of the conic section. For matrix $A$, which is assumed to be positive definite, the conic section is an ellipse. It is then clear that the simultaneous diagonalization of $A$ and $B$ can be realized by making, one after the other, the following three linear transformations: i) a rotation $R_1$, which carries the coordinate axes over the symmetry axes of the ellipse associated to $A$; ii) a dilatation $D$ of the new coordinate axes, which changes the ellipse into a circle of radius one (with this transformation, non orthogonal, the conic section associated to $B$ changes its symmetry axes, but it remains a conic section centered in the origin); iii) a rotation $R_2$, which carries the coordinate axes over the symmetry axes of the second conic section (while the circle remains a circle of radius one). The overall transformation $U = R_2 D R_1$ diagonalizes simultaneously $A$ and $B$, and even more, it changes $A$ into the identity matrix. The generalization to $n > 2$ is obvious.

### 2.7.5 Linearization and stability

From the uncoupled equations (2.7.10), it clearly appears that, *for the linearized problem,* the origin is a stable equilibrium point if all roots $\lambda_1, \ldots, \lambda_n$ of the secular equation are positive, and unstable if one at least is zero or negative. As already commented in Chapter 1, the stability properties of the linearized system do not always transfer in a trivial way to the original nonlinear problem. This happens however, for the particular conservative Lagrangian systems we are dealing with, in the most interesting cases:

Proposition 36 *For the nonlinear system,*

*i) if all roots $\lambda_1, \ldots, \lambda_n$ are positive, then the equilibrium is stable;*

*ii) if at least one of the roots is negative, then the equilibrium is unstable.*

The only case in which it is not possible to draw an immediate conclusion, is the (non generic) one in which none of the $\lambda_i$ is negative, but at least one of them is zero. Apart from this case, we can say that the equilibrium point is stable, for the nonlinear as well as for the linear system, if and only if the potential energy has there a minimum. On the other hand, the undecided case is precisely the one in which, looking only at the second derivatives of $V$, it is not possible to establish if $V$ has ha minimum or not. We can then conclude, as anticipated above, that *if we restrict ourselves to the case in which the presence or the absence of a minimum of $V$ is decided by examining the second derivatives of $V$, then the stability condition entering the Lagrange–Dirichlet theorem is not only sufficient, but also necessary.*

Proof. Point i) is an immediate consequence of the Lagrange-Dirichlet theorem: indeed, if all roots are positive, then $V$ certainly has a minimum in the equilibrium configuration, and stability is guaranteed. Concerning point ii), it is a consequence of proposition 10 of Chapter 1 (section 1.4.3). To apply the proposition, we must give the equations of motion the form of a system of $2n$ first order equations, but since the stability properties do not depend on the choice of the coordinates, we are free to choose the most convenient ones, which in our case turn out to be the normal coordinates. In such coordinates the second order equations are the (2.7.10), and the corresponding first order system is

$$\dot{x}_i = v_i \ , \qquad \dot{v}_i = -\lambda_i x_i \ , \qquad i = 1, \dots, n \ .$$

If we order coordinates as $(x_1, v_1, \dots, x_n, v_n)$, then the $2n \times 2n$ Jacobian matrix of the system is

$$\begin{pmatrix} 0 & 1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & 1 \\ & & & & -\lambda_n & 0 \end{pmatrix}$$

and with evidence, its $2n$ eigenvalues are

$$\mu_i^{\pm} = \pm\sqrt{-\lambda_i} \ , \qquad i = 1, \dots, n \ . \tag{2.7.11}$$

As a consequence, if for an index $i$ it is $\lambda_i < 0$, then we have a pair of real opposite eigenvalues $\mu_i^{\pm}$, and the presence of $\mu_i^{+}$, real and positive, implies the instability of equilibrium. □

- As already remarked, proposition 36 is the analog of proposition 10 of Chapter 1, which we also used in the proof of point ii) of proposition 36. One might wonder, however, about point i) of proposition 36, which not only cannot be deduced from proposition 10, but might even appear in conflict, if not with its statement, with some comments we there made. Indeed, from (2.7.11) we see that if all roots $\lambda_i$ are positive, then all eigenvalues $\mu_i^{\pm}$ are purely imaginary, and such a situation was indicated as an uncertain one, while instead point i) of proposition 36 ensures stability. The reason of such an enhanced stability, is that the loss of stability of the nonlinear system, in presence of eigenvalues with vanishing linear part, requires the presence of non conservative forces, which instead, in the conservative Lagrangian context we have restricted, are a priori excluded.

### 2.7.6 Nonlinearity and chaotic motions

Consider a stable equilibrium point, with $\lambda_1, \ldots, \lambda_n > 0$. Although stability, as ensured by proposition 36, persists in the nonlinear system too, the motions of the nonlinear system, if observed for a sufficiently long time, may differ in a significative way from those of the linearized system. Let us refer to normal coordinates, and assume, for simplicity, that forces do not depend on velocities. Then the Lagrangian of the nonlinear system has the form

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{i=1}^{n} L_i(x_i, \dot{x}_i) - V^{\text{int}}(x_1, \ldots, x_n) \ ,$$

with $L_i(x_i, \dot{x}_i) = \frac{1}{2}(\dot{x}_i^2 - \omega_i^2 x_i^2)$, while the interaction potential $V^{\text{int}}$ includes terms at least cubic in $\mathbf{x}$, which couple the otherwise independent harmonic oscillators forming the linearized system. Although small for motions of small amplitude, such terms can nevertheless produce, on long times, important energy exchanges among oscillators. There is indeed no reason why the individual energies $E_i = \frac{1}{2}(\dot{x}_i^2 + \omega_i^2 x_i^2)$ are conserved, and correspondingly, motions of the nonlinear system can can get substantially more complicated.

An interesting example, quite important also hystorically because it was the first one in which chaotic motions have been observed, is provided by the so-called *Hénon–Heiles model* (1964).[26] This is an apparently simple system, consisting of two harmonic oscillators of equal frequency, which in an adapted time unit can be set equal to one, coupled by a cubic interaction potential of the form[27] $V^{\text{int}} = x_1^2 x_2 - \frac{1}{3}x_2^3$; the Lagrangian is then

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}(x_1^2 + x_2^2) - x_1^2 x_2 + \frac{1}{3}x_2^3 \ ,$$

and the corresponding nonlinear equations of motion are

$$\ddot{x}_1 = -x_1 - 2x_1 x_2 \ , \qquad \ddot{x}_2 = -x_2 + x_1^2 - x_2^2 \ . \tag{2.7.12}$$

In the uncoupled linear model ($V^{\text{int}} = 0$), there are two constants of motion: the total energy and the energy of any of the two oscillators. In the nonlinear model, the total energy is certainly conserved as well, and the question naturally arises whether there still esists a second conserved quantity, that is a convenient function $F(x_1, x_2, \dot{x}_1, \dot{x}_2)$ that ramains constant along motions.

Hénon and Heiles solved numerically the equations of motion (2.7.12), and represented motions by means of a suitable *Poincaré section.* We already introduced the notion of Poincaré section

[26] The motivation, which we cannot go into, comes from Celestial Mechanics, more precisely from stellar dynamics: in a certain approximation, $x_1$ and $x_2$ represent the deviations of the motion of a star from a circular motion, in a cylindrically symmetric galaxy. To understand the distribution of stars in a galaxy, it is important to know the number of constants of motion in the problem.

[27] If $V^{\text{int}}$ is homegeneous of degree $2 + s$ (here $s = 1$), then the rescaling $\mathbf{x} = \varepsilon\tilde{\mathbf{x}}$, followed by the (always allowed) division of the Lagrangian by $\varepsilon^2$, gives a new Lagrangian of the form

$$\tilde{L}(\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}}) = \sum_{i=1}^{n} L_i(\tilde{x}_i, \dot{\tilde{x}}_i) - \varepsilon^s V^{\text{int}}(\tilde{x}_1, \ldots, \tilde{x}_n) \ ,$$

identical to the previous one but for the coefficient $\varepsilon^s$ in front of $V^{\text{int}}$. Motions that were previously confined in a neighborhood of radius $\varepsilon$ of the equilibrium point, and correspondingly had small energy $E = \mathcal{O}(\varepsilon^2)$, now take place in a neighborhood of radius one, and have energy $\tilde{E} = E/\varepsilon^2$ of order one. So, considering motions of small energy $E = \varepsilon^2\tilde{E}$ is *exactly* the same as considering motions with energy $\tilde{E}$ and small interaction term $\varepsilon^s V^{\text{int}}$. If $V^{\text{int}}$ is not homogeneous, but is the sum of homogeneous terms $V^{(2+s)}$ of degree $2 + s$, then the interaction term in the rescaled system is $\sum_s \varepsilon^s V^{(2+s)}$.

in Chapter 1, in connection with the forced pendulum. Here we follow the same idea: the four-dimensional space of states is sectioned with a convenient "surface" $\Sigma$ of dimension three; a good choice here is the plane $x_1 = 0$. For any initial datum on such a plane, to be definite with $\dot{x} > 0$, consider the subsequent intersections of the trajectory with $\Sigma$, with $\dot{x} > 0$. Due to energy conservation, these cannot stay anywhere on $\Sigma$, but must satisfy the condition

$$\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + x_2^2) - \frac{1}{3}x_2^3 = E$$

(we already kept into account that $x_1 = 0$). We then see that coordinates $x_2, \dot{x}_2$ are sufficient to locate the intersection point, $\dot{x}_1$ being determined by

$$\dot{x}_1 = \sqrt{2E - \dot{x}_2^2 - x_2^2 + \tfrac{2}{3}x_2^3}\ . \tag{2.7.13}$$

So, the subsequent intersections can be represented graphically on an ordinary plane, the Cartesian plane $x_2, \dot{x}_2$. For any value $E$ of the energy, a map $\Phi_E : \mathbb{R}^2 \to \mathbb{R}^2$ is then defined, namely the map that sends each intersection into the next one. For any $E$, the radicand in (2.7.13) must be positive, so the map is defined in the domain $\mathcal{D}_E$ given by

$$\frac{1}{2}\dot{x}_2^2 + \frac{1}{2}x_2^2 - \tfrac{1}{3}x_2^3 < E\ ;$$

$\mathcal{D}_E$ is easily seen to be a bounded domain (actually an egg-shaped one, elongated the direction of positive $x_2$) for $E \le 1/6$, unbounded instead for $E > 1/6$.

Let us denote $z = (x_2, \dot{x}_2)$. As for the forced pendulum, to each motion of the system there corresponds, for the map, a discrete trajectory, that is a sequence $z_1, z_2, \ldots$ with $z_{k+1} = \Phi_E(z_k)$. If the motion is periodic, then the trajectory for the map is composed by a finite sequence of points, that repeats. If in the original system there is, in addition to $E$, a further constant of motion $F(x_1, x_2, \dot{x}_2, \dot{x}_2)$, then for the map $\Phi_E$, too, there is a constant of motion, mamely

$$G_E(x_2, \dot{x}_2) = F(0, x_2, \dot{x}_1(E, x_2, \dot{x}_2), \dot{x}_2)\ ,$$

with $\dot{x}_1(E, x_2, \dot{x}_2)$ defined by (2.7.13); as a consequence, the subsequent $z_k$ are confined to a level curve $G_E(x_2, \dot{x}_2) = \text{const}$. If instead there are no constants of motion besides $E$, then the intersections $z_k$ can wander on a two-dimensional region of $\mathcal{D}_E$.

Figure 2.8, panel (a), shows the Poincaré section for a small value of $E$, namely $E = 0.08$. The different curves (actually resolved in points) correspond to different trajectories of the map. The clear impression is that a constant of motion $G_E$ does exist, the curves appearing in the figure being the level curves of $G$. It is however enough to raise a little the energy (this means, looking at motions of larger amplitude, for which nonlinear terms in the equations of motion are relatively larger), to observe a drastically different picture. Panel (b) of figure 2.8 shows the Poincaré section of the system for $E = 0.125$: we can see that for some initial data the sequence of intersections are still aligned in well defined invariant curves, as if there was a constant of motion $G_E$. But for other initial data, subsequent intersections appear to fill a definitely two-dimensional region of the domain: *all points that in panel (b) of figure 2.8 do not appear to stay aligned on curves, belong to a single trajectory.* Such a region is commonly called the *chaotic region,* while the region filled by invariant curves is called the *regular* (or ordered) *region.* By increasing $E$, the chaotic region becomes dominant, as is shown by panel (c) of the figure, which refers to $E = 0.1666$, and by panel (d), where a rough estimate of the selative size of the regular region is reported vs. $E$.

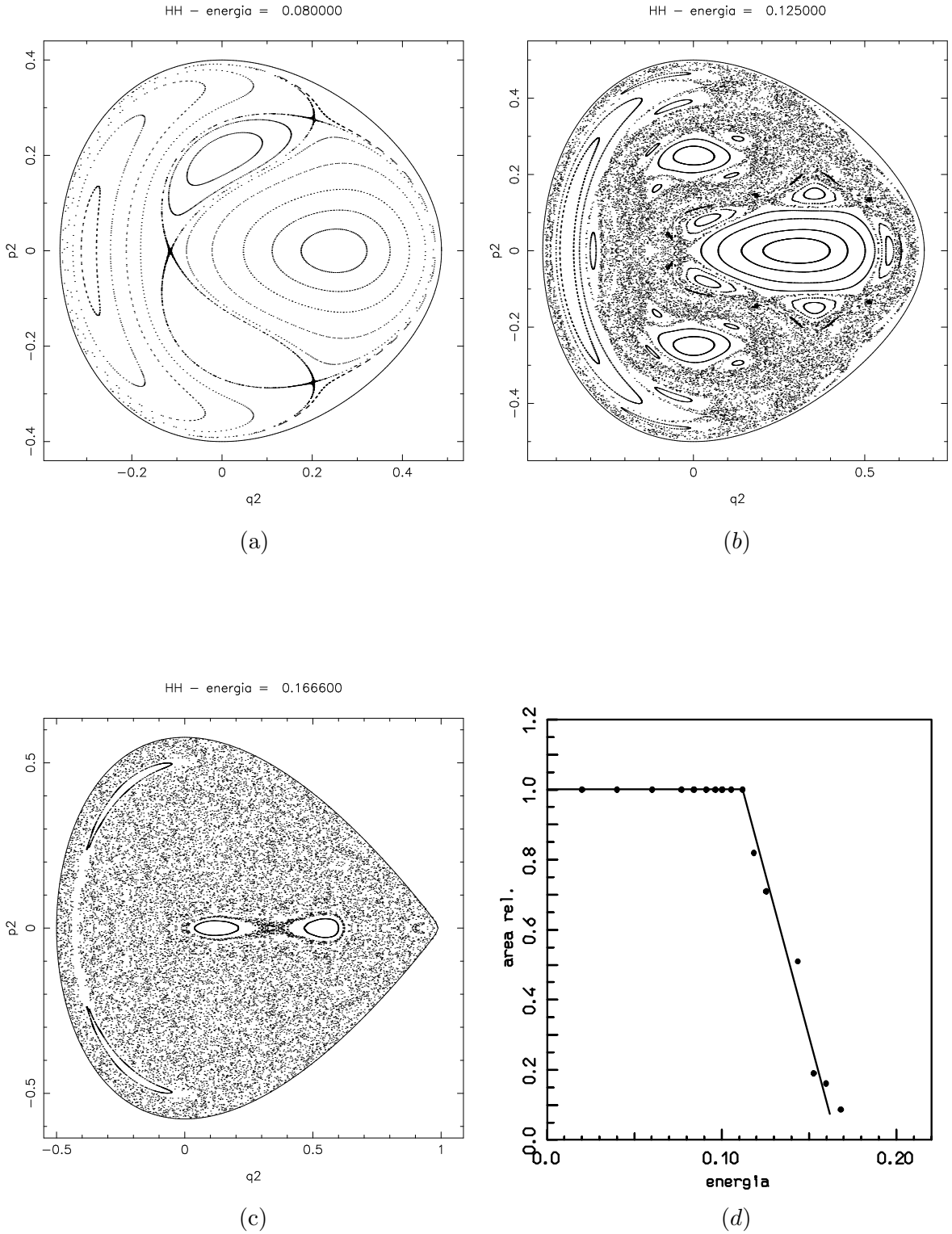


Figure 2.8: *The Poincaré section of the Hénon–Heiles system, for* $E = 0.08$ *(a),* $E = 0.125$ *(b),* $E = 0.1666$ *(c); a rough estimate of the size of the regular region (d).*

The phenomenology, on the whole, is rather similar to that of the forced pendulum and of the Standard map, that we discussed in Chapter 1, section 1.6.1. Here too, the chaotic region is characterized by the presence of exponential divergence of nearby trajectories, more precisely by the fact that the Lyapunov exponent $\chi_{\max}$, in such a region, is positive.

## 2.8 The variational principles of mechanics

The way we followed so far to investigate the laws of motion of a system of point masses, could be called the "differential way", in the sense that we assumed, as the starting point, Newton's differential equation $m\mathbf{a} = \mathbf{F}$, and from it we deduced, still in the form of differential equations, the general laws of motion of a mechanical system (the Lagrange equations). The central idea, underlying the discussion we have done till now, is that true or natural motions of a mechanical system are, among all those we could conceive, the ones that satisfy, point by point, a certain differential relation, characteristic of the system.

In this section we will explore a different way, though in fact equivalent, to formulate the general laws of mechanics, in which the true motion distinguishes, among the conceivable ones, for a global integral property, in the same sense the straight line distinguishes, among all curves of the plane, as the shortest one between two given points, or the trajectory of a ray of light, in a medium with variable refraction index, distinguishes (Fermat's principle) by the fact it minimizes the travel time between two assigned points.

This new way, more geometric, can be called the "variational way", as it refers to that sector of mathematical analysis called variational calculus; the point of arrival are the so-called *variational principles* of mechanics, which bear the name "principles" precisely because from them, taken as principles, the whole mechanics can be deduced. A further reason to be interested in the variational formulation of mechanics is that it is especially suitable for passing from classical mechanics to relativistic theories, where global geometric aspects have great importance. It is not hazardous to say that the variational formulation of any problem is, at least in the language, the most geernal one, and possibly also the deepest one.

The next three subsections are devoted to a quick introduction to variational calculus, which is not assumed to be known, so as to frame our study of the variational principles of mechanics in the general context of variational calculus. *Our study here will not be as rigorous as the subject would require,* and occasionally we shall proceed by intuition, on the basis of examples; for a more complete and rigorous treatment, we demand to dedicated textbooks.

### 2.8.1 Functionals

The elementary problem from which variational calculus originates, is the search for maxima and minima, more generally for stationarity points, of ordinary functions of several real variables. As is well known, for any regular function $F : \mathbb{R}^n \to \mathbb{R}$, the condition for which $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is a stationary point, is that the differential of $F$ vanishes in $\mathbf{x}$, or equivalently $\frac{\partial F}{\partial x_i}(\mathbf{x}) = 0$, $i = 1, \ldots, n$.

We could mention other variational problems for functions defined in $\mathbb{R}^n$ (or in an open domain of $\mathbb{R}^n$), like the search for constrained maxima or minima: *variational calculus* however, in the proper sense, begins when the domain of $F$ is not $\mathbb{R}^n$, nor any other finite-dimensional space, but a space of functions. Consider a set $U$ of functions, for example the set $U_{a,b}$ of (regular) real functions defined in the interval $[a, b]$.

DEFINITION 14 *An application $F : U \to \mathbb{R}$, which associates to any function $u \in U$ a real number, is said to be a* functional *with domain $U$.*

A common notation for functionals is $F[u]$. Simple examples of functionals defined in $U_{a,b}$ are the average

$$F[u] = \frac{1}{b-a} \int_a^b u(x) \mathrm{d}x \ , \tag{2.8.1}$$

any norm like the Euclidean norm or the sup-norm, respectively

$$F[u] = \Big( \int_a^b u^2(x) \mathrm{d}x \Big)^{1/2} \ , \qquad F[u] = \max_{x \in [a,b]} |u(x)| \ , \tag{2.8.2}$$

or the value of $u$, or of its derivative $u'$, in any given point $\bar{x}$:

$$F[u] = u(\bar{x}) \ , \qquad F[u] = u'(\bar{x}) \ . \tag{2.8.3}$$

Let us remark that in the notion of functional, the function $u$ replaces the $n$–tuple $(x_1, \dots, x_n)$, while the continuous variable $x \in [a, b]$, at argument of $u$, replaces the discrete index $i = 1, \dots, n$ which labels the arguments of $F$.

Let us now assume that the domain $U$, as the above example $U_{a,b}$, is a linear vector space (so that inside $U$ it is meaningful to sum functions and to multiply functions by a real number).

DEFINITION 15 *A functional $F$ on $U$ is said to be linear, if for any $u_1, u_2 \in U$ and $c_1, c_2 \in \mathbb{R}$ it is $F[c_1 u_1 + c_2 u_2] = c_1 F[u_1] + c_2 F[u_2]$.*

The average and the examples in (2.8.3) are linear, the examples in (2.8.2) are clearly not.

A functional of great interest in geometry and in mechanics is the one giving the length of a curve. Consider the simple case of curves in the $xy$ plane, which are the graph of a function, that is are of the form $y = u(x)$, $a \le x \le b$. The *length functional,* defined in $U_{a,b}$, is then

$$F[u] = \int_a^b \sqrt{1 + u'^{\,2}(x)} \ \mathrm{d}x \ . \tag{2.8.4}$$

If instead, for example to study curves which go around the origin, we use polar coordinates $r$ and $\vartheta$, and write the curve in the form $r = u(\vartheta)$, $\vartheta_0 \le \vartheta \le \vartheta_1$, then the length functional, defined in $U_{\vartheta_0, \vartheta_1}$, has the form, as is immediately checked,

$$F[u] = \int_{\vartheta_0}^{\vartheta_1} \sqrt{u^2(\vartheta) + u'^2(\vartheta)} \ \mathrm{d}\vartheta \ .$$

Other expressions are found for other systems of coordinates, or for non flat surfaces.

EXERCISE 37 *Consider a cone of semi-aperture $\beta$; for a generic point $P$ on it, use as coordinates the distance $r$ of $P$ from the vertex and the angle $\varphi$ between the half-plane issuing from the cone axis, passing through $P$, and some assigned half-plane also issuing from the axis (polar spherical coordinates of $P$, with fixed colatitude); see figure 2.9. Write the functional which gives the length of a curve of equation $r = u(\varphi)$, between given $\varphi_0$ and $\varphi_1$.* [Answer: $F[u] = \int_{\varphi_0}^{\varphi_1} [u^2(\varphi) \sin^2 \beta + u'^2(\varphi)]^{\frac{1}{2}} \mathrm{d}\varphi)]$. *Write again the functional, using instead the polar coordinates $r, \vartheta$ of the plane development of the cone.* [Answer: if $r = u(\vartheta)$, then $F[u] = \int_{\vartheta_0}^{\vartheta_1} [u^2(\vartheta) + u'^2(\vartheta)]^{\frac{1}{2}} \mathrm{d}\vartheta$, as for the curves of the plane written in polar coordinates.]

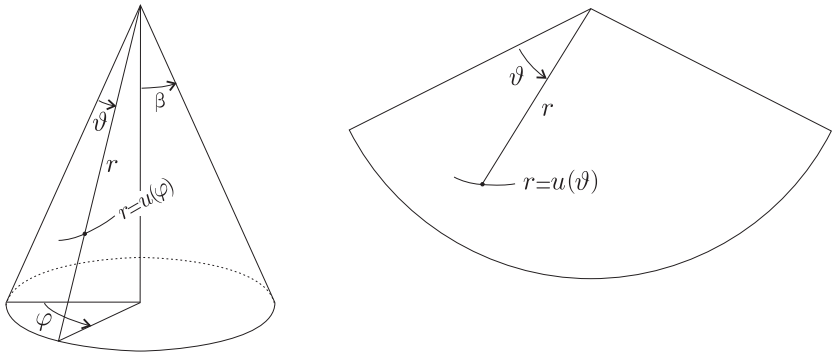


Figure 2.9: *A curve on a cone, represented in two different ways.*

A classical problem is finding, for any surface, the *geodesics,* that is the curves of minimal length (more generally of stationary length, see below) between two given points; these are the curves assumed by a stretched flexible wire, and also (it can be shown) the trajectories of the inertial motion of a point mass confined on the surface.

An important problem in mechanics and optics is computing the travel time on an assigned trajectory $\gamma$ by a point mass, or a ray of light, whose velocity depends in a known way on the position. Restricting ourselves, for simplicity, to planar motions, and to trajectories which are the graphic of a function $y = u(x)$ between two fixed abscissas $a$ and $b$, the travel time is given by

$$T[u] = \int_\gamma \frac{ds}{v} = \int_a^b \frac{\sqrt{1 + u'^2(x)}}{v(x, u(x))}\, \mathrm{d}x \ ,$$

where $v(x, y)$ is the modulus of the velocity in $(x, y)$. In optics, the velocity is directly given by the ratio $c/n(x, y)$, where $c$ is the speed of light in vacuum and $n$ is the refraction index of the medium, so that

$$T[u] = \frac{1}{c} \int_a^b n(x, u(x)) \sqrt{1 + u'^2(x)} \mathrm{d}x \ ; \tag{2.8.5}$$

the *Fermat's principle* states that the effective trajectories are, among the a priori conceivable curves connecting two given points, those for which $T[u]$ is minimal (more generally is stationary). For a point mass on an assigned curve, in absence of active forces $v$ is constant, and the computation of $T$ reduces to the computation of the length of the curve. More generally, if the system is conservative and $V(x, y)$ is its potential energy, the function $v(x, y)$ depends on the energy $E$,

$$v(x, y) = \sqrt{\tfrac{2}{m}(E - V(x, y))} \ .$$

For example, for a point mass that descends along a curve $y = u(x)$ through the origin, starting from the origin with zero velocity ($E = 0$), we have $v(x, y) = \sqrt{2gy}$ (vertical descending $y$ axis), and thus

$$T[u] = \frac{1}{\sqrt{2g}} \int_0^b \sqrt{\frac{1 + u'^2(x)}{u(x)}}\, \mathrm{d}x \ . \tag{2.8.6}$$

One among the very classical problems of mechanics is determining the curve connecting the origin to an assigned point $A$ of the plain, such that the travel time of $P$ on it is minimal; the curve is

called the *brachistocrone* (which means the curve "of the shortest time", from the greek $\beta\rho\alpha\chi\acute{\upsilon}\varsigma$ = short, $\chi\rho\acute{o}\nu o\varsigma$ = time).

We could introduce functionals which depend explicitly on derivatives of $u$ of order higher than one (for example, the functional giving the maximal convexity of a curve), but they are not relevant for the variational principles of mechanics we aim to study, so we will not consider them, and restrict instead our attention to functionals $F[u]$ which depend explicitely on $u$, on its first derivative $u'$, and on the independent variable $x$, as in example (2.8.6).

The notion of functional extends naturally to the case of dependence on two or more functions: for example, the scalar product

$$F[u,v] \,=\, \int_a^b u(x)v(x)\mathrm{d}x$$

is a functional which depends on two functions $u$ e $v$ (and is linear in both of them, or bilinear); the length of a curve in the three-dimensional space, defined by the parametric equations $x = u(t)$, $y = v(t)$, $z = w(t)$, $a \le t \le b$, is the functional

$$F[u,v,w] = \int_a^b \left[u'^2(t) + v'^2(t) + w'^2(t)\right]^{\frac{1}{2}}\mathrm{d}t \ .$$

### 2.8.2 Variation of a functional

The notion of variation of a functional we are going to introduce, is the analog of the notion of directional derivative for functions of a finite number of variables, that we quickly recall.[28]

Le $F$ be a regular function of $n$ real variables, and $\mathbf{u} = (u_1, \ldots, u_n)$ an internal point of its domain of definition $U \subset \mathbb{R}^n$ (the independent variables have been denoted $u_i$, rather than $x_i$, to stress the analogy with functionals). Fix any $n$–tuple $\delta\mathbf{u} = (\delta u_1, \ldots, \delta u_n) \in \mathbb{R}^n$, and consider the values of $F$ in the varied points $\mathbf{u} + \alpha\delta\mathbf{u}$, for real $\alpha$ in a neighborhood of the origin (observe that for small $\alpha$, the varied point certainly belongs to $U$). The directional derivative (or variation) $\delta F$ of function $F$ at point $\mathbf{u}$, relative to the vector (or variation) $\delta\mathbf{u}$, is then defined by

$$\delta F(\mathbf{u}, \delta\mathbf{u}) = \frac{\mathrm{d}}{\mathrm{d}\alpha} F(\mathbf{u} + \alpha\delta\mathbf{u})\Big|_{\alpha=0} \ ;$$

if $F$ is regular, then $\delta F$ is linear in $\delta\mathbf{u}$, as we see by computing the above derivative:

$$\delta F(\mathbf{u}, \delta\mathbf{u}) = \sum_{i=1}^{n} \frac{\partial F}{\partial u_i}(\mathbf{u})\, \delta u_i \ .$$

Making reference to the directional derivative, we can say $F$ is stationary in $\mathbf{u}$, if and only if $\delta F$ vanishes in $\mathbf{u}$ for any choice of the variation $\delta\mathbf{u}$.

Consider now a functional $F$, let $U$ be its domain of definition, and assume $U$ is a linear vector space; the example to have in mind is the domain $U_{a,b}$ we introduced above. Let $u$ be a "point"

[28] The notion of directional derivative, as is known, is somehow poor, and sometimes insufficient to understand the behavior of a function in a neighborhood of a given point, which requires the deeper notion of differential. Critical examples are $F(x,y) = x^2y/[(y-x^2)+y^2]$, or also $F(x,y) = x^3y/[(y-x^2)^2+y^2]$, with in both cases $F(0,0) = 0$. In the first example, $F$ is regular on *all* lines through the origin, but as a function of two variables is not even continuous in the origin; in the second one the function, restricted to *any* line through the origin, is stationary in the origin, but it is not stationary as a function of two variables. Similar pathologies, however, disappear for differentiable functions.

(that is, a function) inside $U$; fix any variation $\delta u$, namely any function belonging to $U$, and consider the one–parameter family of varied functions

$$u(x) + \alpha\delta u(x) \ ,$$

with real $\alpha$ in a neighborhood of the origin. By analogy with the finite–dimensional case, we give the following definition:

DEFINITION 16 *The variation of functional $F$ in $u$, relative to the variation $\delta u$, is the functional*

$$\delta F[u, \delta u] = \frac{\mathrm{d}}{\mathrm{d}\alpha} F[u + \alpha\delta u]\Big|_{\alpha=0} \ .$$

*If $\delta F[u, \delta u]$ esists for any $\delta u$ and is linear in $\delta u$, $F$ is said to be* Gateau–differentiable *in $u$; in turn, $\delta F$ is said to be the* Gateaux–derivative *of $F$.*

It is important to observe, as a support to the definition, that $F[u + \alpha\delta u]$, for any fixed $u$ and $\delta u$, is an ordinary function of the single real variable $\alpha$ (well defined in a neighborhood of zero): so, for functionals too, as in the above example of functions of several variables, we are reduced to the familiar notion of derivative for a function of a single variable.

- Proceeding intuitively, we could avoid the use of the small parameter $\alpha$, and think instead the variation $\delta u$ to be, in a convenient sense, small. It is then spontaneous to define the variation $\delta F$ of the functional, as the linear part in $\delta u$ of the finite increment $\Delta F = F[u + \delta u] - F[u]$; think of a Taylor expansion truncated to the linear term, in which higher order terms in $\delta u$ (or in its derivatives that, as we shall see, naturally enter the calculation of $\delta F$) are disregarded. The directional derivative (reflect on the finite-dimensional case) is precisely a simple procedure to separate the linear part from higher order terms (which contain $\alpha$ at a power larger than one, and thus vanish when, after taking the derivative, we set $\alpha = 0$). Such intuitive considerations could be formalized in a deeper notion of derivative of a functional, called the *Fréchet derivative,* which appropriately transports to functionals the deep notion of differential of a function, rather than the notion of directional derivative. We shall not enter such more complex definition, which is not necessary to our purposes. It is a useful exercise to see, in the next few examples, that the intuitive procedure and the precise notion we gave lead to the same result.

A few examples are useful to practise the notion of variation of a functional. If $F[u]$ is the average (2.8.1), then

$$F[u + \alpha\delta u] = \frac{1}{b-a}\int_a^b (u(x) + \alpha\delta u(x))\mathrm{d}x \ , \qquad \delta F[u, \delta u] = \frac{1}{b-a}\int_a^b \delta u(x)\mathrm{d}x$$

($F$ being linear, the variation $\delta F$ depends only on the variation $\delta u$ and not on $u$; for the same reason, it was not necessary to set $\alpha = 0$ in the calculus of $\delta F$). For $F[u] = \int_a^b u^2(x)\mathrm{d}x$ it is instead

$$\begin{aligned} \delta F[u, \delta u] &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b [u(x) + \alpha\delta u(x)]^2\mathrm{d}x\,\Big|_{\alpha=0} = \int_a^b 2[u(x) + \alpha\delta u(x)]\delta u(x)\mathrm{d}x\,\Big|_{\alpha=0} \\ &= 2\int_a^b u(x)\delta u(x)\mathrm{d}x \end{aligned}$$

(now $\delta F$ also depends on the "point" $u$ at which the variation is computed).

Let us consider now a functional like the length of a curve (2.8.4), or the travel time (2.8.5), (2.8.6), in which the derivative $u'$ of $u$ appears explicitly; more generally, let us consider the important case of functionals $F : U_{a,b} \to \mathbb{R}$, of the form

$$F[u] = \int_a^b L(u(x), u'(x), x)\mathrm{d}x \ , \tag{2.8.7}$$

where $L : \mathbb{R}^3 \to \mathbb{R}$ is a regular function of the three real variables $u$, $u'$ and $x$. So, for the length functional (2.8.4) it is $L(u, u', x) = (1 + u'^2)^{1/2}$ ($L$ depends in fact only on $u'$); for the functional (2.8.6) it is instead $L(u, u', x) = (2g)^{-1/2}[(1 + u'^2)/u]^{1/2}$.

Computing the variation of a generic functional of the form (2.8.7), is not difficult: indeed, we have

$$F[u + \alpha\delta u] = \int_a^b L(u(x) + \alpha\delta u(x), u'(x) + \alpha\delta u'(x), x)\,\mathrm{d}x \ ,$$

where $\delta u'$ denotes the derivative of $\delta u$ with respect to $x$; by applying the definition we then find

$$\begin{aligned} \delta F[u, \delta u] &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b L\big(u(x) + \alpha\delta u(x), u'(x) + \alpha\delta u'(x), x\big)\,\mathrm{d}x \,\Big|_{\alpha=0} \\ &= \int_a^b \Big[\frac{\partial L}{\partial u}(u(x), u'(x), x)\,\delta u(x) + \frac{\partial L}{\partial u'}(u(x), u'(x), x)\,\delta u'(x)\Big]\,\mathrm{d}x \ . \end{aligned}$$

With an integration by parts, we can eliminate $\delta u' = \frac{\mathrm{d}}{\mathrm{d}x}\delta u$, and conclude with the following proposition:

Proposition 37 *The variation of the functional (2.8.7) is*

$$\delta F[u, \delta u] = \frac{\partial L}{\partial u'}\delta u\Big]_a^b - \int_a^b \Big(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}\Big)\,\delta u\,\mathrm{d}x \ .$$

(to make the expression more readable, we omitted the arguments of $L$ and the argument $x$ in $\delta u$; it is suggested to write in detail the complete expression).

In many cases, sensible problems require we restrict the attention to functions $u$ with fixed estremes, that is funtions such that $u(a) = u_a$, $u(b) = u_b$, with assigned $u_a, u_b$; correspondingly, we must restrict ourselves to variations which vanish at the extremes, $\delta u(a) = \delta u(b) = 0$. If we denote

$$U_{a,b}^{A,B} = \big\{u \in U_{a,b}; u(a) = A, u(b) = B\big\} \ ,$$

then $u \in U_{a,b}^{u_a,u_b}$ and $\delta u \in U_{a,b}^{0,0}$. Problems like the search of a geodesic, or of the brachistochone, *between two fixed end points,* as we discussed above, obviously require a similar restriction. For variations that vanish at the extremes, it is clearly

$$\delta F[u, \delta u] = -\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}\right)\delta u(x)\,\mathrm{d}x \ . \tag{2.8.8}$$

Exercise 38 *Write the variation of the functional expressing the length of a curve in Cartesian coordinates,* $F[u] = \int_a^b [1 + u'^2(x)]^{1/2}\mathrm{d}x$.

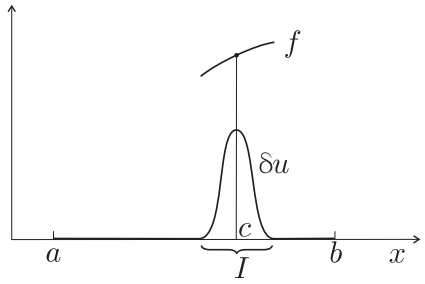


Figure 2.10: *To illustrate the proof of proposition 38.*

### 2.8.3 Stationariety of a functional and Euler–Lagrange equation

Let us restrict the attention to functionals of the form (2.8.7), defined in a domain $U_{a,b}^{u_a,u_b}$ with fixed extremes, and correspondingly, to variations in $U_{a,b}^{0,0}$, that is $\delta u(a) = \delta u(b) = 0$. The following definition is spontaneous:

DEFINITION 17 *A functional $F$ of the form (2.8.7) in $U_{a,b}^{u_a,u_b}$ is said to be stationary in $u$, and correspondingly $u$ is said to be a stationarity point for $F$, if $\delta F[u, \delta u] = 0$ for any choice of $\delta u \in U_{a,b}^{0,0}$.*

The following easy but fundamental lemma holds:

PROPOSIZIONE 38 *Consider a functional $F$ of the form (2.8.7), defined in $U_{a,b}^{u_a,u_b}$; $F$ is stationary in $u$, if and only if $u$ satisfies the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u} = 0 \ .$$

Such an equation is called the *Euler–Lagrange equation* associated to the function $L$.

PROOF. Consider the expression (2.8.8) of $\delta F$. Quite trivially, if $u$ satisfies the Euler–Lagrange equation, then $\delta F$ vanishes for any choice of $\delta u$. Conversely, let us suppose $F$ is stationary in $u$, and show that the quantity

$$f = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u} \ ,$$

which for any given $u$ inside $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}$ is an ordinary function of $x$, identically vanishes in $[a, b]$. Indeed, assume by absurd $f \neq 0$ at some point $c \in [a, b]$. By continuity, there exists a neighborhood $I$ of $c$ (a right or left neighborhood if, respectively, $c = a$ or $c = b$) where $f$ has constant sign, for example is positive. Then by choosing a particular regular variation $\delta u$, with $\delta u(x) = 0$ for $x \notin I$ and $\delta u(x) > 0$ inside $I$ (if $I = (x_0, x_1)$, a possible choice is $\delta u(x) = (x - x_0)^3(x_1 - x)^3$ inside $I$), we obtain, by integration, $\delta F > 0$, against the assumption $\delta F[u, \delta u] = 0$ for any $\delta u$ (see figure 2.10). Therefore, $f$ identically vanishes in $[a, b]$.[29] □

[29] What we did, is generalizing to the case of a continuous "index" $x$ the property, obvious for a discrete index $i$, that if $\sum_i f_i g_i = 0$ for any $n$-tuple $(g_1, \ldots, g_n)$, then $f_i = 0$ for any $i$. In the discrete case the very obvious procedure is taking $n$ tuples $(g_1, \ldots, g_n)$ of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$, so as to "filter" the single components $f_i$, and show one by one they vanish. Here, with continuous $x$, although we could not take $\delta u$ different from zero only in one point, we followed exactly the same idea.

So, the stationarity of $F = \int_a^b L(u(x), u'(x), x)\mathrm{d}x$ for arbitrary variations, vanishing at the extremes, is seen to be equivalent to a differential equation for the stationarity "point" $u$, namely to the Euler–Lagrange equation associated to $L(u, u', x)$, with the condition of fixed extremes $u(a)$, $u(b)$.[30]

For example, if the functional $F$ represents the lenght of a curve in the plane, then we have

$$L(u, u', x) = \sqrt{1 + u'^{\,2}}\ , \tag{2.8.9}$$

and the Euler–Lagrange equation writes

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{u'}{\sqrt{1 + u'^{\,2}}} = 0\ ;$$

by computing the derivative we find $u''(1+u'^{\,2})^{-3/2} = 0$, and thus $u'' = 0$. The equation is solved by straight lines. This is not surprising, nevertheless it is a proof that straight lines are the geodesics of the plane, that is the two elementary notions of line—the curve with constant slope (differential notion) and the curve of the stretched wire (variational notion)—do coincide.

- Take the Lagrangian (2.8.9). Even without writing its Euler–Lagrange equation, we can observe that $u$ does not appear explicitly in $L$, and consequently its "conjugated momentum" $p = \frac{\partial L}{\partial u'}$ must be conserved (that is, must be constant in $x$). We find immediately $p = u'/\sqrt{1 + u'^{\,2}}$, and correspondingly deduce $u'$ itself is constant—another way to see $u$ represents a line. A further possibility: $L$ does not depend explicitly on the independent variable $x$; consequently, the quantity $E(u, u') = u'\frac{\partial L}{\partial u'} - L$ (though in no way does it represent an energy) must be constant. We easily find $E = -1/\sqrt{1 + u'^{\,2}}$, which also leads to $u'$ being constant.

Exercise 39 *Prove the lines are the geodesics of the plain, using polar coordinates $(r, \vartheta)$; assume $r = u(\vartheta)$.* [Answere: The Lagrangian is $L(u, u') = \sqrt{u^2 + u'^2}$, and the corresponding Euler–Lagrange equation is $uu'' - 2u'^2 - u^2 = 0$. The equation is solved by the function which expresses lines in polar coordinates, that is $r = a/\sin(\vartheta - \varphi)$, where $a > 0$ is the distance of the line from the origin and $\varphi$ is its inclination with respect to the polar axis].

Exercise 40 *Prove the geodesics of the cylinder and of the cone appear as lines in the planar development of the surfaces. Find the condition on the aperture of the cone, such that a geodesic can have a double point (that is: a lace, with a single point fixed on the conical surface, can stay stretched on the surface without slipping off the tip).* [Hint: think of the plane development of the cone, obtained by cutting the cone along the generatrix passing through the double point.]

Exercise 41 *Verify that the arcs of a great circle are the geodesics of the sphere* (it is enough to check it for the equator, or a meridian).

Exercise 42 *Write the equation of the brachistochrone, and show it is solved by the cycloid.* [Look, for the solution, at Appendix E, where another nice property of the cycloid is studied: a point mass constrained on a cycloid, and subject to gravity ("cycloidal pendulum"), at variance with the usual circular pendulum, is *exactly* isochronous.]

[30] The associated *problem,* however, is not the classical initial values Cauchy problem, that we would have if $u(a)$ and $u'(a)$, instead of $u(a)$ and $u(b)$, were assigned. Rather, it is the so-called Sturm–Liouville problem, in which the position data at the two extremes are given. At variance with the Cauchy problem, the Sturm–Liouville problem does not always admit a solution, nor in general the solution is unique.

It is not difficult to generalize the above considerations to the case of a functional $F$ which depends not on a single function, but on an $n$-tuple of functions $\mathbf{u} = (u_1, \dots, u_n)$, with $u_h : [a, b] \to \mathbb{R}$, $h = 1, \dots, n$. Indeed, let $L$ be a function: $\mathbb{R}^{2n+1} \to \mathbb{R}$, and consider a functional of the form

$$F[\mathbf{u}] = \int_a^b L(\mathbf{u}(x), \mathbf{u}'(x), x)\mathrm{d}t \ ; \tag{2.8.10}$$

proceeding as above, for any arbitrary variation $\delta\mathbf{u} : [a, b] \to \mathbb{R}^n$, we define the variation $\delta F[\mathbf{u}, \delta\mathbf{u}]$ as a directional derivative, that is

$$\delta F[\mathbf{u}, \delta\mathbf{u}] = \frac{\mathrm{d}}{\mathrm{d}\alpha} F[\mathbf{u} + \alpha\delta\mathbf{u}]\Big|_{\alpha=0} \ .$$

By computing the derivative we get

$$\delta F[\mathbf{u}, \delta\mathbf{u}] = \sum_{h=1}^{n} \frac{\partial L}{\partial u'_h} \delta u_h \Big]_a^b - \int_a^b \sum_{h=1}^{n} \Big( \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial L}{\partial u'_h} - \frac{\partial L}{\partial u_h} \Big) \delta u_h \mathrm{d}x \ .$$

If we restrict ourselves to fixed extremes, and correspondingly to variations which vanish at the extremes, we finally get

$$\delta F[\mathbf{u}, \delta\mathbf{u}] = - \int_a^b \sum_{h=1}^{n} \Big( \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial L}{\partial u'_h} - \frac{\partial L}{\partial u_h} \Big) \delta u_h \mathrm{d}x \ .$$

With obvious generalization of the notion of stationarity of a functional, proposition 38 generalizes into

Proposition 38' *Consider a functional $F$ of the form (2.8.10); $F$ is stationary in $u$, with the restriction of fixed extremes, if and only if $u_1, \dots, u_n$ satisfy the Euler–Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial L}{\partial u'_h} - \frac{\partial L}{\partial u_h} = 0 \ , \qquad h = 1, \dots, n \ .$$

Proof. Proceeding as for proposition 38, we easily see that each of the $n$ functions $f_i = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial u'_i} - \frac{\partial L}{\partial u_i}$ must vanish. To show $f_i$ vanishes, it is enough to take $\delta u_j = 0$ for $j \neq i$ and $\delta u_i$ as in the proof of proposition 38. □

### 2.8.4 The Hamilton principle

The Hamilton principle, actually the simplest among the variational principles of mechanics, is nothing but the transposition to Lagrangian mechanics of what we learned above, with no change at all but language and notation.

Consider a Lagrangian system with $n$ degrees of freedom, of Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, and for any movement[31] $\mathbf{q}(t)$ between two fixed times $t_0$ and $t_1$, denote by $S[\mathbf{q}]$ the functional

$$S[\mathbf{q}] = \int_{t_0}^{t_1} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\mathrm{d}t \ ;$$

[31] With little abuse, we are here denoting the movement (a function of $t$) with the same symbol of the variable $\mathbf{q}$ in the configuration space; the improperty is useful, since it makes the notation lighter, but attention should be payed not to get confused.

$S$ is called the *Hamilton integral,* or also the *Hamiltonian action.* By translating into the language of mechanics what we already learned, we can say that, for variations $\delta\mathbf{q}(t)$ vanishing at the extremes, the variation of the functional is

$$\delta S[\mathbf{q},\delta\mathbf{q}] = -\int_{t_0}^{t_1}\Big[\sum_{h=1}^{n}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_h}-\frac{\partial L}{\partial q_h}\Big)\delta q_h\Big]\mathrm{d}t\ ;$$

correspondingly the following proposition, known as the *Hamilton principle,* does hold:

Proposition 39 *The movement $\mathbf{q}(t)$, $t\in[t_0,t_1]$, makes stationary the action functional $S$, with the restriction of variations $\delta\mathbf{q}(t)$ vanishing at the extremes, if and only if it is a natural movement, that is it solves the Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_h}-\frac{\partial L}{\partial q_h}=0\ ,\qquad h=1,\dots,n\ .$$

So, the Hamilton principle represents the variational formulation of the laws of mechanics.

It is interesting to see how simple, and natural, is deducing from the Hamilton principle the two invariance properties of the Lagrange equations, that we discussed in section 2.4.6. Consider first the invariance of the equations for changes of coordinates, and recall that for any change of coordinates (diffeomorphism) $\mathbf{q}=\mathbf{q}(\tilde{\mathbf{q}},t)$, $\tilde{L}$ is defined by replacement of variables:

$$\tilde{L}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}},t)=L(\mathbf{q}(\tilde{\mathbf{q}},t),\dot{\mathbf{q}}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}},t),t)\ .$$

Then for movements $\mathbf{q}(t)$, $\tilde{\mathbf{q}}(t)$ that correspond to each other, at any time $t$ it is

$$\tilde{L}(\tilde{\mathbf{q}}(t),\dot{\tilde{\mathbf{q}}}(t),t)=L(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\ ,$$

and thus

$$\tilde{S}[\tilde{\mathbf{q}}]=\int_{t_0}^{t_1}\tilde{L}(\tilde{\mathbf{q}}(t),\dot{\tilde{\mathbf{q}}}(t),t)\mathrm{d}t=\int_{t_0}^{t_1}L(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\mathrm{d}t=S[\mathbf{q}]\ .$$

As a consequence, one integral is stationary if and only if the other is. Since, as we have seen, the stationariety of the integrals is equivalent to their respective Lagrange equations, we conclude that the Lagrange equations in the new variables are satisfied, if and only if they are satisfied in the old variables.

It is similarly easy and natural to deduce the gauge invariance. Indeed, let

$$S[\mathbf{q}]=\int_{t_0}^{t_1}L(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\mathrm{d}t\ ,\qquad S'[\mathbf{q}]=\int_{t_0}^{t_1}\Big[L(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)+L_0(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\Big]\mathrm{d}t\ ,$$

with $L_0=\frac{\mathrm{d}F}{\mathrm{d}t}$. With evidence, it is

$$S'[\mathbf{q}]=S[\mathbf{q}]+F(\mathbf{q}(t_1),t_1)-F(\mathbf{q}(t_0),t_0)\ ;$$

so for fixes extremes (just apply the definition of variation), although $S'[\mathbf{q}]\neq S[\mathbf{q}]$, we get $\delta S'=\delta S$. The conclusion is immediate.

# APPENDICES

## A Examples of non-holonomic constraints

A simple example of a non-holonomic constraint is the "ski which does not drift". The configuration of a ski which is free to slide on a plain, can be identified by three coordinates: the position of a given point, say the point $P$ at the center of the binding, and the angle $\varphi$ the ski forms with a fixed direction, for example the $x$-axis (figure 2.11). By saying the ski does not drift, we mean the velocity $\mathbf{v} = \dot{x}\,\mathbf{e}_x + \dot{y}\,\mathbf{e}_y$ of $P$ is necessarily parallel to the ski, that is to the versor $\mathbf{e} = \cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y$; it is then $\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$, or equivalently in differential form

$$\sin\varphi \mathrm{d}x - \cos\varphi \mathrm{d}y = 0 \ .$$

By comparing with a generic differential form in the three variables $x, y, \varphi$

$$f_x(x,y,\varphi)\mathrm{d}x + f_y(x,y,\varphi)\mathrm{d}y + f_\varphi(x,y,\varphi)\mathrm{d}\varphi \ = \ 0 \ ,$$

we see that

$$f_x = \sin\varphi \ , \qquad f_y = -\cos\varphi \ , \qquad f_\varphi = 0 \ .$$

It is easy to verify that the closure conditions are not satisfied, no matter how the "integrating factor" $\lambda(x,y,\varphi)$ is chosen: indeed if we pose

$$g_x = \lambda(x,y,\varphi) f_x \ , \qquad g_y = \lambda(x,y,\varphi) f_y \ , \qquad g_\varphi = \lambda(x,y,\varphi) f_\varphi \ ,$$

it is $g_\varphi = 0$, and consequently the closure conditions $\frac{\partial g_x}{\partial \varphi} = \frac{\partial g_\varphi}{\partial x}$, $\frac{\partial g_y}{\partial \varphi} = \frac{\partial g_\varphi}{\partial y}$, lead to the system

$$\lambda\cos\varphi + \frac{\partial\lambda}{\partial\varphi}\sin\varphi = 0 \ , \qquad -\lambda\sin\varphi + \frac{\partial\lambda}{\partial\varphi}\cos\varphi = 0 \ ,$$

which is homogeneous and solved only by $\lambda = 0$.

In this example we made, as an exercise, a detailed analytic treatment. But the fact the constraint is non-holonomic can be seen *a priori,* in a quite immediate way: take any configuration $(x,y,\varphi)$; with evidence, starting from it we are able to reach any other configuration $(x',y',\varphi')$, by performing only displacement which respect the constraint: for example, we can (i) rotate the ski, orienting it towards $(x',y')$; then (ii) reach $(x',y')$ with velocity parallel to the ski; finally (iii) rotate again the ski, so as it assumes the desired orientation $\varphi'$. This excludes there exists $F$ (non-trivial) such that $F(x',y',\varphi') = F(x,y,\varphi)$, i.e. the constraint cannot be written in integer form.

Esercizio 43 (The unicycle) *Proof that the constraint of pure rolling, for a wheel on the plain, is not holonomic (assume the wheel stays orthogonal to the plane).* [Hint: identify the configuration of the wheel with the four coordinates $x$, $y$, $\varphi$ and $\vartheta$, where $x, y$ are the coordinates of the wheel hub, $\varphi$ is the angle formed by the plane of the wheel with the $x$ axis, and $\vartheta$ is the angle formed by an assigned radius of the wheel with the vertical direction. The constraint that the contact point between wheel and floor has zero velocity, can be expressed by the two relations, including velocities, $\dot{x} + R\dot{\vartheta}\cos\varphi = 0$ and $\dot{y} + R\dot{\vartheta}\sin\varphi = 0$.]

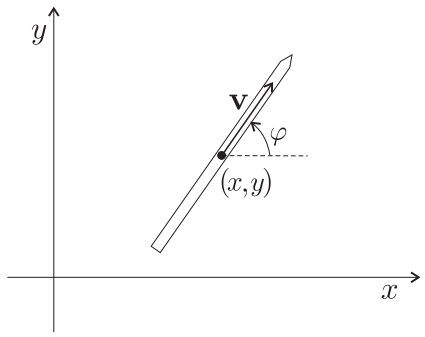


Figure 2.11: *A non-holonomic constraint: the "ski which does not drift".*

# B Cardinal equations, ideality of the constraint and Lagrange equations for a rigid body

The ideality of a constraint, for a system of $N$ point masses $P_1, \dots, P_n$, using the virtual velocities $\mathbf{u}_i$ can be written

$$\sum_{i=1}^{N} \mathbf{\Phi}_i \cdot \mathbf{u}_i = 0 \ ; \tag{B.1}$$

virtual velocities, we recall, are arbitrary velocities compatible with the constraint.

On the other hand, the rigidity constraint implies that for any motion compatible with the constraint, at any time, there exists a vector $\boldsymbol{\omega} \in \mathbb{R}^3$, such that

$$\mathbf{u}_i = \mathbf{u}_O + \boldsymbol{\omega} \times (P_i - O) \ , \tag{B.2}$$

where $O$ is any assigned point, fixed in the body frame: for example, but not necessarily, any of its points. *By varying $\mathbf{u}_O$ and $\boldsymbol{\omega}$, we find all possible choices of the virtual velocities $\mathbf{u}_i$.* (For the justification of such claims, which would require too much space, we demand to any textbook of rational mechanics.)

From these premises, it is easy to deduce that

Proposition 40 *For a rigid body, the ideality of the constraint is equivalent to the pair of equations*

$$\sum_{i=1}^{N} \mathbf{\Phi}_i = 0 \ , \qquad \sum_{i=1}^{N} (P_i - O) \times \mathbf{\Phi}_i = 0 \ . \tag{B.3}$$

Proof. If we substitute expression (B.2) of $\mathbf{u}_i$ inside (B.1), and make a cyclic permutation of vectors in the mixed vector product, we find

$$\mathbf{u}_0 \cdot \sum_{i=1}^{N} \mathbf{\Phi}_i + \boldsymbol{\omega} \cdot \sum_{i=1}^{N} (P_i - O) \times \mathbf{\Phi}_i = 0 \ .$$

For the arbitrariness of $\mathbf{u}_O$ and $\boldsymbol{\omega}$, the conclusion is immediate. □

In turn, equations (B.3) express the fact that the resultant and the resultant moment of the internal forces, inside the system, do vanish, and thus such equations are equivalent to the cardinal

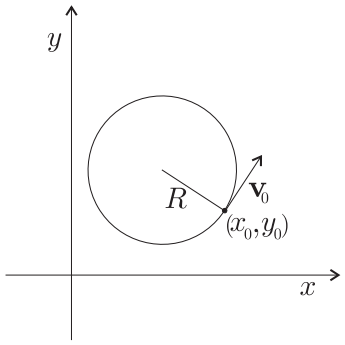


Figure 2.12: *The Larmor motion and its stabilizing effect.*

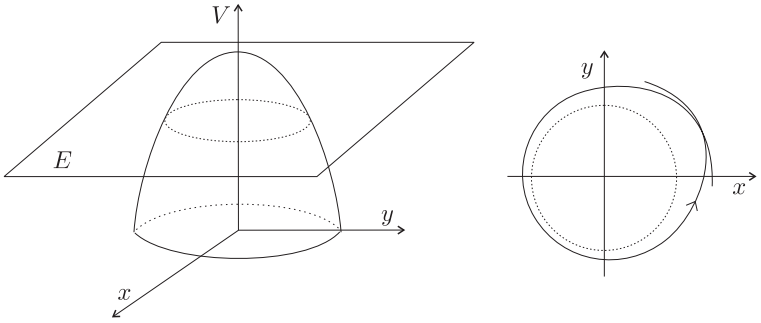


Figure 2.13: *Illustrating the phenomenon of the magnetic stabilization.*

equations (if they are satisfied, then the cardinal equations hold, and conversely). We conclude that *the ideality of the rigidity constraint is equivalent to the cardinal equations.*

Finally, as we have seen in chapter 2.4.1, the assumption of ideality of the constraint, for any holonomic system, implies the Lagrange equations are satisfied: as a consequence, *for a rigid body the cardinal equations imply the Lagrange equations are satisfied, and consequently they univocally determine the motion.* Let us recall that in general, for a non-rigid system, the cardinal equations are satisfied along any motion, and thus are implied by the Lagrange equations, but are not sufficient to determine it. Instead, *for rigid bodies, cardinal equations and Lagrange equations are equivalent.* This implies, in particular, the six scalar equations corresponding to the cardinal equations are independent.

## C The magnetic stabilization

We show here that in some cases, by adding a suitable magnetic field, an unstable equilibrium configuration can become stable.

Consider a two-dimensional harmonic repulsor, that is a point particle of mass $m$ which is free to move in a plain and is subject to a conservative repulsive force $\mathbf{F} = kr\mathbf{e}_r$, with $r^2 = x^2 + y^2$; the system is described by the Lagrangian

$$L_0(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V_0(r) \ , \qquad V_0(r) = -\frac{1}{2}kr^2 \ .$$

Quite evidently, the origin is an unstable equilibrium configuration for any $k \geq 0$ (for $k = 0$, we

have the usual instability of the free particle).

Let us now assume the particle has an electric charge $e$, and is immersed in a constant magnetic field orthogonal to the plain, $\mathbf{B} = B\mathbf{e}_z$; the corresponding Lorentz force is

$$\mathbf{F} = eB\,\mathbf{v} \times \mathbf{e}_z \ .$$

It is not difficult to see that if $B$ is sufficiently large, then the origin gets stable. This is evident, first of all, for $k = 0$: indeed in such a case the particle, as is known, moves uniformly on a circle passing through the initial point $(x_0, y_0)$, tangent to the initial velocity, of radius equal to the Larmor radius

$$R = \frac{mv}{|eB|} \ ,$$

where $v$ is the modulus of the velocity (see figure 2.12). We clearly see that for any $B \neq 0$, if $(x_0, y_0)$ is sufficiently close to the origin and $v$ is sufficiently small, the particle stays arbitrarily close to the origin (and velocity stays close to zero); according to the definition, the origin is a stable equilibrium point. For $k = 0$, of course, the origin is any point of the plane: that is in absence of active forces, but in presence of a magnetic field, all points of the plain are stable equilibrium points.

For $k > 0$, it is not difficult to see that if the magnetic field is sufficiently strong, namely if

$$B^2 \ > \ 4\,\frac{km}{e^2} \ , \tag{C.1}$$

then the origin is stable. A possible way to prove stability is to write explicitly the equations of motion, which are linear and can be solved, and observe stability directly on the solutions. A nicer way, which does not require to solve the equations of motion and indeed does not require any calculation, is the following one: the velocity dependent potential associated to the Lorentz force is, in the present notations,

$$V_1 = \frac{1}{2} e\ \mathbf{B} \times \mathbf{v} \cdot \mathbf{r} \ ,$$

and the Lagrangian is

$$L = K - V_0 - V_1 \ .$$

If we make the position

$$\boldsymbol{\omega} = \frac{e\mathbf{B}}{2m} \ ,$$

and add and subtract to $L$ the quantity $\frac{1}{2}m\omega^2 r^2$, we obtain

$$L = K - W_0 - V_c - V_1 \ ,$$

where

$$W_0(r) = V_0(r) + \frac{1}{2}m\omega^2 r^2 \ , \qquad V_c = -\frac{1}{2}m\omega^2 r^2 \ , \qquad V_1 = m\ \boldsymbol{\omega} \times \mathbf{v} \cdot \mathbf{r} \ .$$

Formally, $L$ is the Lagrangian of a particle initially described by Lagrangian $L_0 = K - W_0$, then observed in a coordinate system which rotates with angular velocity $\boldsymbol{\omega}$ with respect to it. This means the terms $V_c$ and $V_1$ disappear if, formally, we "come back to the initial system", that is in reality we pass from our frame to new frame which rotates with angular velocity $-\boldsymbol{\omega}$. In such a frame the potential energy is $W_0$, and clearly (Lagrange–Dirichlet theorem), if condition (C.1) is satisfied, then the origin is stable. The stability does not change by passing from one system to

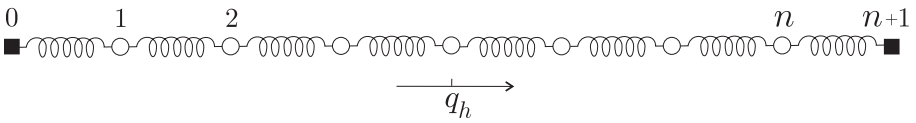


Figure 2.14: *The discrete vibrating string with fixed ends.*

the other ($r$ does not change, $v$ changes by quantities small with $r$). For generic central repulsive potentials, condition (C.1) is replaced by the more general condition

$$\frac{e^2B^2}{4m} > V_0''(0) \ .$$

It is also easy to understand qualitatively how the particle moves: any central potential like $W_0$ produces rosette motions (exceptionally, closed trajectories); in the original system a rotation is added, and motions remain of rosette type.

There is a curious aspect in the magnetic stabilization, which in a sense reveales its fragility: while in general adding a little friction improves stability (in the future), in such a case friction destroys stability. The idea is illustrated in figure 2.13: if the potential $V_0$ has a maximum in the origin, then for any value of $E$, the motion must stay outside the curve of intersection between the plane of constant energy $E$ and the graphic of the potential energy, that is, outside the circle drawn in the figure (simply because $E = K + V_0$ with $K \geq 0$; remember $V_1$ does not contribute to the energy). Now any small friction inexorably consumes energy: correspondingly the plane of constant energy lowers, the circle gets larger, and the particle moves away from the equilibrium point. Such a behavior should not surprise: friction lowers velocity, the magnetic force decreases, its stabilizing effect reduces.

## D The discrete vibrating string

Consider a system of $n$ point masses of equal mass $m$, staying on a line as in figure 2.14; denote by $x_1, \ldots, x_n$ their abscissas, and assume that nearby points interact with a potential energy $\mathcal{V}(r)$, having the typical profile of molecular potentials, as schematically drawn in figure 2.15. The function $\mathcal{V}$ is assumed to have a minimum in $r = a$; a Taylor expansion gives then

$$\mathcal{V}(r) = \text{cost} + \frac{k}{2}(r-a)^2 + \mathcal{O}((r-a)^3) \ ,$$

with $k = \mathcal{V}''(a)$. Let us suppose the chain is closed by fixed ends (or "rigid walls"), that is by two fixed particles in $x_0 = 0$ and $x_{n+1} = \mathcal{L}$; let us assume for simplicity (but it is not really important) $\mathcal{L} = (n+1)a$. In such conditions it is evident that the configuration

$$x_h^* = h\,a \ , \qquad h = 1, \ldots, n \ ,$$

is a minimum of the overall potential energy, and thus is a stable equilibrium configration.

Let us choose as Lagrangian coordinates the displacements from equilibrium $q_h = x_h - x_h^*$. With evidence, the Lagrangian of the system has the form

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\sum_{h=1}^{n} \dot{q}_h^2 + \frac{k}{2}\sum_{h=0}^{n}(q_{h+1} - q_h)^2 + \mathcal{O}(\|\mathbf{q}\|^3) \ ,$$

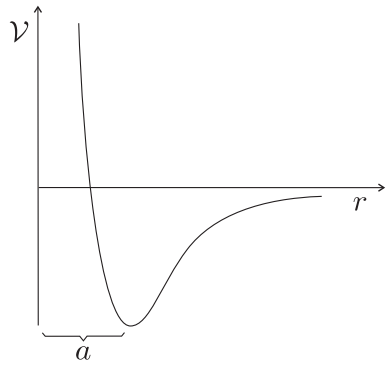


Figure 2.15: *A typical molecular potential.*

where it is understood $q_0 = q_{n+1} = 0$. If we truncate the Lagrangian to its quadratic part and divide it by $m$ (multiplying by a constant does not change the equations of motion), we finally obtain the Lagrangian

$$L^*(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\sum_{h=1}^{n} \dot{q}_h^2 - \frac{\Omega^2}{2}\sum_{h=0}^{n}(q_{h+1} - q_h)^2 \ , \tag{D.1}$$

with $\Omega^2 = k/m$; the corresponding equations of motion are

$$\ddot{q}_h = \Omega^2(q_{h+1} - 2q_h + q_{h-1}) \ , \qquad h = 1, \ldots, n \ . \tag{D.2}$$

Exercise 44 *Verify that even without assuming $x_{n+1} = (n+1)a$, the truncated Lagrangian has the form D.1; what is in this case $\Omega^2$?*

*A. The analogy with the continuous vibrating string.* We can think of growing the number $n$ of points at fixed overall length $\mathcal{L}$, mantaining however constant the density $\mu$, that is posing $m = \mu a$, as well as the elastic constant per unit length $\kappa$, that is posing $k = \kappa/a$. It is then

$$\Omega^2 = \frac{c^2}{a^2} \qquad \text{with} \qquad c^2 = \frac{\kappa}{\mu} \ .$$

Let us observe the constant $c$ has the physical dimension of a velocity.

We now replace the integer index $h$ by the corresponding equilibrium coordinate $x_h^* = ha$, dropping however the asterisk (there is no risk of confusion with the coordinate $x_h$ of the $h$–th point, a symbol that we abandoned); we can then use the notation $q(x_h, t)$ in place of $q_h(t)$. With trivial algebric manipulation, the equation of motion can be rewritten

$$\frac{\partial^2 q}{\partial t^2}(x_h, t) = c^2 \, \frac{\frac{q(x_h+a,t)-q(x_h,t)}{a} - \frac{q(x_h,t)-q(x_h-a,t)}{a}}{a} \ .$$

*Proceeding heuristically,* in the limit $a \to 0$ (at fixed $x$, not $h$) we recognize, at the right hand side, the second partial derivative of $q$ with respect to the $x$; we "deduce" in this way the wave equation for a continuous chain,

$$\frac{\partial^2 q}{\partial t^2} - c^2 \frac{\partial^2 q}{\partial x^2} = 0 \ . \tag{D.3}$$

The deduction is independent on the conditions we imposed at the ends, and is also meaningful (as an heuristic deduction!) for an infinite string. If the string in infinite, as is well known, equation (D.2) is solved by any $q(x,t)$ of the type

$$q(x,t) = f_1(x - ct) + f_2(x + ct) \ ,$$

with any $f_1$, $f_2$; $f_1$ e $f_2$ represent profiles that propagate, respectively, with velocity $c$ and $-c$.

For a string with fixed ends, that is $q(0,t) = q(\mathcal{L},t) = 0$ for any $t$, (D.3) can be solved by recalling that an orthonormal basis for functions $[0,\mathcal{L}] \to \mathbb{R}$ which vanish at the extremes, is provided by

$$u^{(j)}(x) = C \, \sin \frac{j\pi x}{\mathcal{L}} \ , \qquad 1 \leq j < \infty \ , \qquad C = \sqrt{\frac{2}{\mathcal{L}}} \ ; \tag{D.4}$$

orthonormal means

$$\int_0^{\mathcal{L}} u^{(j)}(x) \, u^{(l)}(x) \, \mathrm{d}x = \delta_{jl} \ .$$

Any solution $q(x,t)$ can then be written in the form

$$q(x,t) = \sum_{j=1}^{\infty} \xi_j(t) u^{(j)}(x) \ .$$

By substituting such an expression in the equation of motion (D.3), we find

$$\sum_{j=1}^{\infty} (\ddot{\xi}_j + \omega_j^2 \xi_j) \sin \frac{j\pi x}{\mathcal{L}} = 0 \ ,$$

with

$$\omega_j = \frac{jc\pi}{\mathcal{L}} \ . \tag{D.5}$$

Each coefficient of the series must vanish, so each coefficient $\xi_j$ must satisfy the equation

$$\ddot{\xi}_j + \omega_j^2 \xi_j = 0 \ , \qquad 1 \leq j < \infty \ ,$$

namely that of the harmonic oscillator. As a conclusion, the general solution of the string with fixed ends can be written

$$q(x,t) = \sum_j A_j \cos(\omega_j t + \varphi_j) \, u^{(j)}(x) \ .$$

It is interesting to observe that all angular frequencies $\omega_j$ are multiples of a fundamental one $\omega_1$:

$$\omega_j = j \, \omega_1 \ .$$

*B. Normal modes of the discrete chain.* Let us go back to the discrete chain, with Lagrangian (D.1) and equations of motion (D.2). We are dealing with a system with many degrees of freedom, so to find the normal modes of oscillation we cannot apply the general method, which would require solving an algebric equation of degree $n$. Proceeding however in analogy with the continuous case, it is not difficult to see that $u^{(j)}(x)$ in (D.4) should be replaced by the corresponding discrete quantities

$$u_h^{(j)} = C \, \sin \frac{hj\pi}{n+1} \ , \qquad 1 \leq j \leq n \ , \qquad C = \sqrt{\frac{2}{n+1}} \ .$$

The $n$ vectors $\mathbf{u}^{(j)} = (u_1^{(j)}, \ldots, u_n^{(j)})$ defined in this way are orthonormal: indeed,

LEMMA 41 *For any $j$ and $l$, $1 \le j, l \le n$, it is*

$$\mathbf{u}^{(j)} \cdot \mathbf{u}^{(l)} = \delta_{jl} \ .$$

PROOF. We must show that

$$\sum_{h=1}^{n} \sin \frac{hj\pi}{n+1} \sin \frac{hl\pi}{n+1} = \frac{n+1}{2}\, \delta_{jl} \ . \tag{D.6}$$

Elementary trigonometric identities give the left hand side the expression

$$\frac{1}{2} \sum_{h=1}^{n} \Big[ \cos \frac{h(j-l)\pi}{n+1} - \cos \frac{h(j+l)\pi}{n+1} \Big] \ . \tag{D.7}$$

(a) Case $j+l$ and $j-l$ odd: posing $m = j+l$ or $j-l$, we find

$$\cos \frac{hm\pi}{n+1} = -\cos \frac{(n+1-h)m\pi}{n+1} \ ,$$

so the terms in the sum cancel out two by two; the central one with $h = (n+1)/2$, which does exist for odd $n$, also vanishes.
(b) Case $j+l$ and $j-l$ even: it is convenient to start the sums in (D.6) and (D.7) from zero (zeros are added). Posing $r = m/2$, with $m$ as above, it is evidently

$$\sum_{h=0}^{n} \cos \frac{hm\pi}{n+1} = \mathrm{Re} \sum_{h=0}^{n} e^{2\pi i h r/(n+1)} \ .$$

But for $r \neq 0$, the sum at the right hand side runs for an integer number of times on the roots of unity (it runs once, on the $(n+1)$th roots, if $r$ and $n+1$ are relatively primes, otherwise, if $s$ is the common divisor, it runs $s$ times on the $\frac{n+1}{s}$th roots), so it vanishes. For $m = 0$ the sum is instead $n+1$. This happens only in the first term in (D.7), when $j = l$; the conclusion follows. □

It is now spontaneous to introduce new coordinates $\xi_1, \ldots, \xi_n$, actually the normal coordinates which decouple the equations of motion, by posing

$$q_h = \sum_{j=1}^{n} \xi_j \, u_h^{(j)} = C \sum_{j=1}^{n} \xi_j \, \sin \frac{hj\pi}{n+1} \ . \tag{D.8}$$

If we substitute this expression inside the equations of motion (D.2), we find

$$C \sum_{j=1}^{n} \ddot{\xi}_j \sin \frac{hj\pi}{n+1} = C \sum_{j=1}^{n} \xi_j \Big( \sin \frac{(h+1)j\pi}{n+1} - 2 \sin \frac{hj\pi}{n+1} + \sin \frac{(h-1)j\pi}{n+1} \Big) \ ,$$

and since

$$\sin \frac{(h+1)j\pi}{n+1} + \sin \frac{(h-1)j\pi}{n+1} = 2 \sin \frac{hj\pi}{n+1} \cos \frac{j\pi}{n+1} \ ,$$

the right hand side assumes the form

$$2C \sum_{j=1}^{n} \xi_j \sin \frac{hj\pi}{n+1} \Big( \cos \frac{j\pi}{n+1} - 1 \Big) \ .$$

Using the identity

$$\cos\frac{j\pi}{n+1} - 1 = -2\sin^2\frac{j\pi}{2(n+1)} \ ,$$

we then obtain for the equations of motion, the expression

$$\sum_{j=1}^{n}(\ddot{\xi}_j + \omega_j^2\xi_j)\, u_h^{(j)} = 0 \ , \qquad h = 1, \dots, n \ , \tag{D.9}$$

having denoted

$$\omega_j = 2\Omega \sin\frac{j\pi}{2(n+1)} \ . \tag{D.10}$$

From (D.9), which is perhaps more clear in the vector notation $\sum_{j=1}^{n}(\ddot{\xi}_j + \omega_j^2\xi_j)\mathbf{u}^{(j)} = 0$, thanks to the linear independence of the $n$ vectors $\mathbf{u}^{(j)}$ (which for lemma 41 are indeed orthogonal), we obtain for each $\xi_j$ the equation of the harmonic oscillator:

$$\ddot{\xi}_j + \omega_j^2\xi_j = 0 \ , \qquad j = 1, \dots, n \ .$$

The general solution of the system is then

$$q_h(t) = \sum_{j=1}^{n} A_j \cos(\omega_j t + \varphi_j)\, \sin\frac{hj\pi}{n+1} \ .$$

- It is interesting to compare (D.10) with (D.5). It clearly appears that for any fixed $j$ and large $n$, it is

  $$\omega_j = j\omega_1 + \mathcal{O}((j/n)^3) \ ,$$

  and also

  $$\omega_j = j\frac{\Omega\pi}{n+1} + \mathcal{O}((j/n)^3) = \frac{jc\pi}{\mathcal{L}} + \mathcal{O}((j/n)^3) \ .$$

  Therefore, as was to be expected, in the limit $n \to \infty$ expressions (D.10) and (D.5) do coincide.

- With evidence, (D.8) is a discrete Fourier transform; in vector notation it can be written

  $$\mathbf{q} = U\boldsymbol{\xi} \ , \qquad U_{hj} = C\sin\frac{hj\pi}{n+1} \ .$$

  Thanks to lemma 41 and to the simmetry of matrix $U$, it is easy to see that the inverse of $U$ is $U$ itself, that it the transformation is involutory.

# E The brachistochrone

As discussed in section 2.8.1, a point mass which descends from the origin to a point of abscissa $b$, along an assigned curve $y = u(x)$, takes a time

$$T[u] = \frac{1}{\sqrt{2g}}\int_0^b L(u(x), u'(x))\,\mathrm{d}x \ ,$$

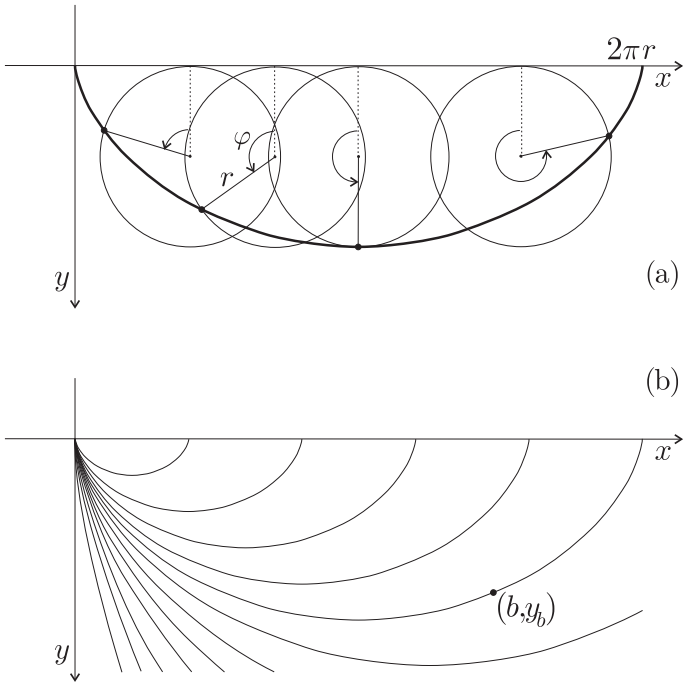


Figure 2.16: *La cicloide (a); una famiglia di cicloidi con diverso $r$, passanti per l'origine (b).*

where

$$L(u,u') = \sqrt{\frac{1+u'^{\,2}}{u}} \ .$$

The curve that minimizes the functional $T$ between the fixed extremes $(0,0)$ and $(b,y_b)$ is called the *brachistocrone.*

To find such a curve, let us first determine the Euler–Lagrange equation associated to $T$. From the expression of $L$ we have

$$\begin{aligned} \frac{\partial L}{\partial u} &= -\frac{1}{2L}\,\frac{1+u'^{\,2}}{u^2} = -\frac{L}{2u} \ , \qquad \frac{\partial L}{\partial u'} = \frac{u'}{Lu} \ , \\ \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} &= \frac{Luu'' - u'(Lu' + u\frac{\mathrm{d}L}{\mathrm{d}x})}{L^2u^2} \ , \end{aligned}$$

and using $\frac{\mathrm{d}L}{\mathrm{d}x} = u'\frac{\partial L}{\partial u} + u''\frac{\partial L}{\partial u'}$ we also get, after a few simplifications,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} = \frac{u'' - \frac{1}{2}L^2u'^2}{L^3u^2} \ .$$

We then easily obtain for the Euler–Lagrange equation the expression

$$2uu'' + u'^{\,2} + 1 = 0 \ .$$

It is not difficult to verify the equation is solved by the *cycloid.* The cycloid is the plane curve described by an assigned point at the border of a wheel, which rolls without slipping on a linear

guide (figure 2.16a). It is easy to provide a parametric description $x(\varphi)$, $y(\varphi)$ of the curve, namely (look at the figure)

$$x(\varphi) = a + r(\varphi - \sin\varphi) \ , \qquad y(\varphi) = r(1 - \cos\varphi) \ , \qquad 0 \le \varphi \le 2\pi \ ;$$

for $a = 0$ the curve passes, as we wish, through the origin. From the parametric equations we obtain $u'$ and $u''$ as functions of $\varphi$: namely

$$u' = \frac{\frac{\mathrm{d}y}{\mathrm{d}\varphi}}{\frac{\mathrm{d}x}{\mathrm{d}\varphi}} = \frac{\sin\varphi}{1 - \cos\varphi}$$

(observe $u' \to \infty$ for $\varphi \to 0, \pi$), and

$$u'' = \frac{\frac{\mathrm{d}u'}{\mathrm{d}\varphi}}{\frac{\mathrm{d}x}{\mathrm{d}\varphi}} = -\frac{1}{r(1 - \cos\varphi)^2} \ .$$

It is then immediate to verify that $2uu'' + u'^{\,2} + 1$ vanishes identically in $\varphi$, that is, the cycloid solves the Euler–Lagrange equation. It is worthwhile to observe that the parametric equations of the cycloid contain two arbitrary constants $a$ e $r$, which can be used to fit the boundary conditions. From the requirement the curve passes through the origin we deduce, as already remarked, $a = 0$; $r$ is instead determined by imposing the curve passes through $(b, y_b)$. If we trace a family of cycloids with $a = 0$ and different $r$, we easily see that, in this problem, there is a solution for any $y_b \ge 0$ (figure 2.16b).

What we did is not completely satisfying: we checked the cycloid solves the Euler–Lagrange equation, but we did not really construct the solution starting from the equation. The best way to do is to exploit the property that, since $L$ does not depend explicitly on $x$, then the quantity $E = u' \frac{\partial L}{\partial u'} - L$ (though not having the meaning of an energy) is conserved. Computation gives

$$E(x) = -\frac{1}{\sqrt{C(x)}} \ , \qquad \text{with} \quad C(x) = u(x)(1 + u'(x)^2) \ ;$$

$C$ itself is evidently constant for any solution $u$, and it is convenient to use it in place of $E$.[32] For any choice of the value $c$ of the constant, the expression $C(x) = c$ is a differential equation of the first order, precisely $u(1 + u'^{\,2}) = c$, that is

$$u' = \pm\sqrt{\frac{c - u}{u}} \ .$$

By separating the variables we obtain, for example for the sign '+',

$$\mathrm{d}x = \sqrt{\frac{u}{c - u}}\,\mathrm{d}u \ , \qquad x = a + \int_0^u \sqrt{\frac{v}{c - v}}\,\mathrm{d}v \ ;$$

the integration gives

$$x(u) = a + c\arctan\sqrt{\frac{u}{c - u}} - \sqrt{u(c - u)} \ , \tag{E.1}$$

[32] It is an useful exercise to check that $C(x)$ is effectively constant. Indeed,

$$C' = u'(1 + u'^{\,2}) + 2uu'u'' = u'(1 + u'^{\,2} + 2uu'') = 0 \ .$$

and by inversion we obtain, in principle, the solution $u(x)$ we were searching for. The inversion cannot be written explicitly, but it is spontaneous to introduce the parameter

$$\vartheta = \arctan \sqrt{\frac{u}{c-u}} \ .$$

Indeed from the very definition of $\vartheta$ we get $(c-u)\tan^2\vartheta = u$, which in turn gives

$$u = \frac{c\tan^2\vartheta}{1+\tan^2\vartheta} = c\sin^2\vartheta \ ;$$

from (E.1) it then follows

$$x = a + c\vartheta - c\sqrt{\sin^2\vartheta(1-\sin^2\vartheta)} = a + c(\vartheta - \sin\vartheta\cos\vartheta) \ .$$

Passing to $\varphi = 2\vartheta$ we finally get

$$x = a + \frac{c}{2}(\varphi - \sin\varphi) \ , \qquad u = \frac{c}{2}(1-\cos\varphi) \ ,$$

which are the parametric equations of the cycloid with $r = c/2$.

**The cycloidal pendulum**

The cycloid has several interesting geometric and mechanical properties. One of them is that the *cycloidal pendulum,* that is the problem of a point mass constrained to a cycloid, subject to gravity, is *exactly* isochronous: the period of oscillation (at variance with respect to the usual circular pendulum) is independent of the amplitude.

It is convenient to represent the curve with the point of minimum in the origin, as in figure 2.17a (the wheel rolls on the line $y = 2r$; the $y$ axis points upwards). If $s$ denotes the arc length from the origin, then the Lagrangian of the cycloidal pendulum[33] is

$$L(s,\dot s) = \frac{1}{2}m\dot s^2 - mgy(s) \ ,$$

with $y(s)$ to be determined. For a generic curve, we would not be able to write $y(s)$ explicitly, but for the cycloid we can do:

Lemma 42 *For the cycloid it is*

$$y(s) = \frac{s^2}{8r} \ . \tag{E.2}$$

The proof is below. The isochronicity of the cycloidal pendulum immediately follows: using (E.2), the Lagrangian becomes

$$L(s,\dot s) = \frac{1}{2}m\dot s^2 - \frac{1}{2}\,\frac{mg}{4r}\,s^2 \ ,$$

that is the Lagrangian of a harmonic osillator with angular frequency

$$\omega = \sqrt{\frac{g}{4r}} \ .$$

This means the motion is *exactly* harmonic.

[33] Which has nothing to do with the Lagrangian of the above considered Euler–Lagrange problem!

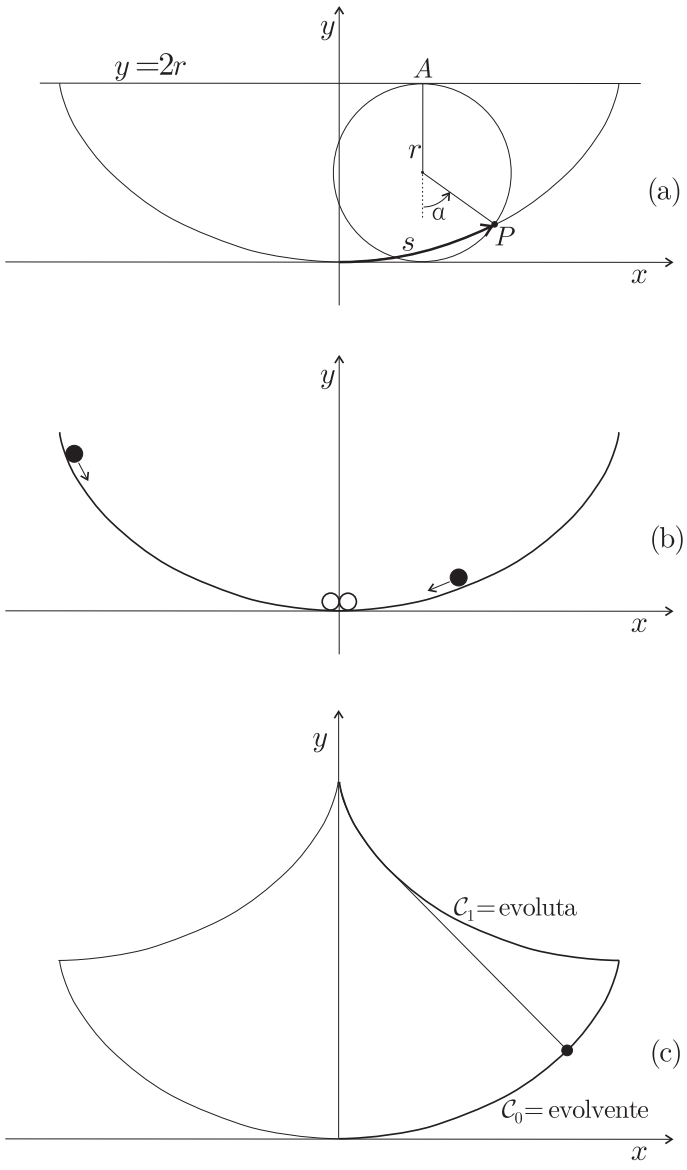


Figure 2.17: *The cycloidal pendulum.*

- A curious consequence is this: whatever is the position on which we leave the point mass free to move (with zero initial velocity), the time necessary to reach the origin is a fourth of period, that is exactly $\frac{1}{2}\pi\omega = \pi\sqrt{r/g}$, regardless of the starting point. So, two masses started simultaneously (with zero velocity) from positions like in figure 2.17b, will always collide in the origin.

- A cycloidal pendulum can be practically constructed by realizing a platform as smooth as possible, having the profile of a cycloid. But there is a better way, illustrated in figure 2.17c: we can modify the trajectory of a pendulum, by letting the wire lean against a conveniently profiled wall, that is a convenient curve $\mathcal{C}_1$. A little reflection shows that to generate in such a way a curve $\mathcal{C}_0$ — the cycloid or another one — $\mathcal{C}_1$ must be the locus of the centers of curvature of $\mathcal{C}_0$. The curve $\mathcal{C}_1$ is called the *evolute* of $\mathcal{C}_0$, and conversely, $\mathcal{C}_0$ is called the *evolvent* of $\mathcal{C}_1$ (in the limit situation of the circle, which has constant center of curvature, the evolute degenerates in a point). A nice property of the cycloid, that we will not prove (though it would not be difficult), is that evolvent and evolute do coincide: precisely, if the wall against which the wire leans has a cycloidal profile, and the length of the wire is $4r$, an identical cycloid is obtained. By exploiting this property, it is not difficult to realize a cycloidal pendulum with much smaller friction.

Proof of the Lemma. It is convenient to write the parametric equations of the cycloid, making reference to the coordinates $x, y$ and to the angle $\alpha$ represented in figure 2.17. The equations are

$$x = r(\alpha + \sin\alpha) \ , \qquad y = r(1 - \cos\alpha) \ ;$$

from them it follows

$$\mathrm{d}x = r(1 + \cos\alpha)\mathrm{d}\alpha \ , \qquad \mathrm{d}y = r\sin\alpha\mathrm{d}\alpha \ ,$$

and thus

$$\mathrm{d}s^2 = r[(1 + \cos\alpha)^2 + \sin^2\alpha]\ \mathrm{d}\alpha^2 = 2r(1 + \cos\alpha)\ \mathrm{d}\alpha^2 = 2\,\frac{1 + \cos\alpha}{\sin^2\alpha}\ \mathrm{d}y \ .$$

But

$$\frac{1 + \cos\alpha}{\sin^2\alpha} = \frac{1}{1 - \cos\alpha} = \frac{r}{y} \ ,$$

and so

$$\mathrm{d}s = \sqrt{2r}\ \frac{\mathrm{d}y}{\sqrt{y}} = \mathrm{d}\sqrt{8ry} \ ;$$

taking into account that $s = 0$ for $y = 0$, we deduce $s = \sqrt{8ry}$, which gives (E.2). □