1) QUESTIONS ABOUT PREVIOUS EXERCISES

2) QUICK REVIEW ABOUT FUNDAMENTAL LIMITS (AND THEOREMS)

3) EXERCISES

$$\lim_{n\to+\infty} x_n = x_0 \begin{array}{l} \nearrow +\infty \\ \to \ell\in\mathbb{R} \\ \searrow 0 \end{array} \qquad \lim_{x\to x_0} f(x) = \lim_{n\to+\infty} f(x_n) = f(x_0)$$

Notes:

$n, x \in \mathbb{R}$

$\log x = \log_e x \, (= \ln x)$ $\qquad$ $(\text{Log}\, x = \log_{10} x)$

$\text{tg}\, x = \tan x$

$\arcsin x = \sin^{-1}(x) \neq \sinh(x)$

$$L = \lim_{x\to x_0} f(x)g(x) \quad \begin{cases} \lim\limits_{x\to x_0} f(x) = \ell \in \mathbb{R} \\ \lim\limits_{x\to x_0} g(x) = m \in \mathbb{R} \end{cases} \longrightarrow L = \ell \cdot m$$

$$L = \lim_{x\to x_0} f(x) \qquad g, h \text{ s.t. } \forall x \; g(x) \leq f(x) \leq h(x)$$

$$\begin{cases} \lim\limits_{x\to x_0} g(x) = \ell \\ \lim\limits_{x\to x_0} h(x) = \ell \end{cases} \longrightarrow L = \ell$$

$$L = \lim_{x\to+\infty} \frac{\sin x}{x} = \lim_{x\to+\infty} \underset{\uparrow \atop \lim = 0}{\left(\frac{1}{x}\right)} \; \sin x \qquad L = 0$$

lim doesn't exist

Know $\sin x$ is limited $\forall x$ $\qquad -1 \leq \sin x \leq 1$

$g(x) = -\frac{1}{x}$, $h(x) = \frac{1}{x}$ $\qquad \lim g(x) = \lim h(x) = 0$

$$L = \lim_{x\to x_0} g(f(x)) \quad \begin{cases} \lim\limits_{x\to x_0} f(x) = y_0 \\ \lim\limits_{y\to y_0} g(y) = \ell \end{cases} \longrightarrow L = \ell$$

$$\lim_{y\to y_0} g(y)$$

$$L = \lim_{x\to 0} \cos(\sin x^2) \underset{y = x^2}{=} \lim_{y\to 0} \cos(\sin y) \underset{z = \sin y}{=} \lim_{z\to 0} \cos z = 1$$

$$L=\lim_{x\to0}\cos(\sin x^2)=\lim_{y\to0}\cos(\sin y)=\lim_{z\to0}\cos z=1$$

$y=x^2$, $x\to0$, $y\to0$

$z=\sin y$, $y\to0$, $z\to0$

$$L=\lim_{w\to w_0}f(w)^{g(w)}\qquad\begin{cases}\lim\limits_{w\to w_0}f(w)=\ell\\ \lim\limits_{w\to w_0}g(w)=m\end{cases}\longrightarrow \text{NOT TRUE}\quad \xcancel{L=\ell^m}$$

$2^w,\ w^3$

$$L=\lim_{x\to0^+}(\sin x^2)^{\frac{1}{\log x^2}}=\lim_{x\to0^+}e^{\log\left[(\sin x^2)^{\frac{1}{\log x^2}}\right]}$$

$$=\lim_{x\to0^+}e^{\frac{1}{\underbrace{\log \underbrace{x^2}_{0}}_{-\infty}}\cdot\underbrace{\log \underbrace{\sin x^2}_{0}}_{-\infty}}$$

$$\lim_{x\to0^+}\frac{\log\sin x^2}{\log x^2}=\lim_{\substack{y\to0^+\\ y=x^2\\ x\to0^+,\ y\to0^+}}\frac{\log\left(\frac{\sin y}{y}\cdot y\right)}{\log y}$$

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

$$=\lim_{y\to0^+}\frac{\log\frac{\sin y}{y}+\log y}{\log y}=\lim_{y\to0^+}\frac{\overbrace{\log}^{0}\frac{\sin y}{y}\xrightarrow{}1}{\underbrace{\log y}_{-\infty}}\to0+1=0+1$$

$$\lim_{x\to0^+}e^{\frac{1}{\log x^2}\log\sin x^2}\left(=e^{\lim\limits_{x\to0^+}\frac{1}{\log x^2}\log\sin x^2}\right)=e^1=\boxed{e}$$

FUNDAMENTAL LIMITS

$$\boxed{\lim_{x\to0}\frac{\sin x}{x}=1},\quad \lim_{x\to0}\frac{\sin^{-1}x}{x}=1,\quad \lim_{x\to0}\frac{\sinh x}{x}=1$$

$$\lim_{x\to0}\frac{\operatorname{tg}x}{x}=1,\quad \lim_{x\to0}\frac{\operatorname{tg}^{-1}x}{x}=1,\quad \lim_{x\to0}\frac{\tanh x}{x}=1$$

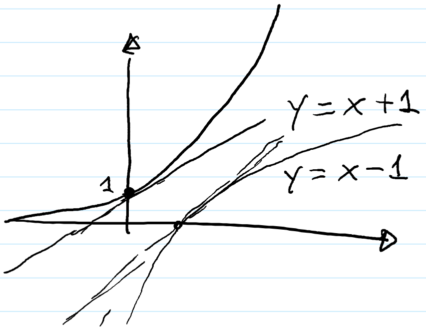
$$\lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2},\quad \lim_{x\to0}\frac{\cosh x-1}{x^2}=\frac{1}{2}$$

$$\lim_{x\to0}\left(1+\frac{1}{x}\right)^x=e$$

$$\lim_{x \to 0} \left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$



$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

$$\downarrow$$

$$\frac{\cancel{1}+\alpha x \cancel{-1}+o(x)}{x}$$

$$\alpha + o(1) \to \alpha$$

$\alpha = 2 \quad (1+x)^2 = 1+2x+x^2$

$\alpha = 3 \quad (1+x)^3 = 1+3x+ \ldots$

$\to (1+x)^\alpha = 1+\alpha x+\underbrace{\ldots \ldots}$

$x^\beta, x \quad \beta > 1 \quad \lim_{x \to 0} \frac{x}{x^\beta} = 0$

$x^\beta = o(x) \to$

$$(1+x)^2 = 1+2x+\underbrace{x^2}_{o(x)} = 1+2x+o(x) \simeq 1+2x$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$\sin x \sim x$

$\hookrightarrow \quad \sin x = x + o(x)$

5) $$\lim_{n \to 0} \frac{\overset{\to 0}{n} - \overset{\to 0}{\sin(3n)}}{\underset{\to 0}{n} - \underset{\to 0}{\sin(2n)}} \quad \left[\frac{0-0}{0-0} = \frac{0}{0}\right]$$

$$\lim_{n \to 0} \frac{n - \frac{\sin 3n}{3n} \cdot 3n}{n - \frac{\sin 2n}{2n} \cdot 2n} = \lim_{n \to 0} \frac{\cancel{n}\left(1 - 3\overset{\to 1}{\boxed{\frac{\sin 3n}{3n}}}\right)}{\cancel{n}\left(1 - 2\underset{\to 1}{\boxed{\frac{\sin 2n}{2n}}}\right)} = \frac{1-3}{1-2} = 2$$

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7) $$\lim_{h\to 0} \frac{\sin(h^2-h)}{h} \cdot \frac{h^2-h}{h^2-h}$$

$$= \lim_{h\to 0} \underbrace{\left(\frac{\sin(h^2-h)}{h^2-h}\right)}_{\to 1} (h-1) = 1\cdot(0-1) = -1$$

8) $$\lim_{h\to 0} \frac{1+\sin h-\cos h}{1-\sin h-\cos h} = \lim_{h\to 0} \frac{\overset{\to 1}{\left(\frac{\sin h}{h}\right)}\cdot h + \overset{\to 1/2}{\left(\frac{1-\cos h}{h^2}\right)}\cdot h^2}{-\underbrace{\left(\frac{\sin h}{h}\right)}_{\to 1} h + \underbrace{\left(\frac{1-\cos h}{h^2}\right)}_{\to 1/2} h^2} \approx \lim_{h\to 0} \frac{h+\frac{1}{2}h^2}{-h+\frac{1}{2}h^2}$$

$$\approx \lim_{h\to 0} \frac{h}{-h} = -1$$

12) $$\lim_{h\to 0} \frac{\cos h - \overset{\cos^2 h - \sin^2 x}{\cos 2h}}{1-\cos h} = \lim_{h\to 0} \frac{\cos h - \overset{\cos^2 h = 1-\sin^2 h}{\cos^2 h} + \sin^2 h}{1-\cos h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\cos h - 1}{1-\cos h}}_{-1} + 2 \frac{\sin^2 h}{1-\cos h} \cdot \frac{h^2}{h^2} \qquad \frac{\sin^2 h}{h^2} \to 1 \qquad \frac{h^2}{1-\cos h} \to \frac{1}{1/2} = 2$$

$$= \lim_{h\to 0} -1 + 2(2\cdot 1) = 3$$

13) $$\lim_{h\to 0} \frac{\log(\cos h)}{\sqrt{1+h^2}-1} = \lim_{h\to 0} \frac{\log(1+(-1+\cos h))}{\sqrt{1+h^2}-1} \cdot \frac{\cos h - 1}{\cos h - 1} \qquad \frac{\log(1+(-1+\cos h))}{\cos h - 1} \to 1$$

$$\approx \lim_{h\to 0} \frac{\cos h - 1}{\sqrt{1+h^2}-1} \cdot \frac{h^2}{h^2} \approx \lim_{h\to 0} -\frac{h^2}{\sqrt{1+h^2}-1} \cdot \frac{1}{2} \qquad \frac{\cos h - 1}{h^2} \to -\frac{1}{2}$$

$$\approx \lim_{h\to 0} -\frac{1}{2} \frac{\cancel{h}}{\cancel{h}} \underbrace{\frac{h}{\sqrt{1+\underbrace{\frac{1}{h^2}}_{\to 0}} - \underbrace{\frac{1}{h}}_{\to 0}}}_{\to 1} \qquad h \to 0 \qquad = 0$$

16) $$\lim_{h\to -1} \frac{h+1}{\sqrt[4]{h+17}-2} = \lim_{t\to 2} \frac{\overset{(t^2+4)(t^2-4)}{t^4-16}}{t-2} = \lim_{t\to 2} \frac{\cancel{(t-2)}(t+2)(t^2+4)}{\cancel{t-2}}$$

$t = \sqrt[4]{h+17}$

$h = t^4 - 17$

$h \to -1,\ t \to 2$

$$= 4\cdot 8 = 32$$

22) $$\lim_{x\to 0} \frac{e^x - 1 + \sin x}{(1+x)^2 - 1 + \operatorname{tg} x} = \lim_{x\to 0} \frac{\cancel{x}\left(\overset{\to 1}{\frac{e^x-1}{x}} + \frac{\sin x}{x}\right)}{\cancel{x}\left(1+2x+x^2-1+\operatorname{tg} x\right)} \qquad \frac{\sin x}{x} \to 1$$

$$\lim_{x\to 0}\frac{}{(1+x)^2-1+\operatorname{tg}x} = \lim_{x\to 0}\frac{}{\cancel{x}\left(\underbrace{\frac{1+2x+x^2-1}{x}}+\underbrace{\frac{\operatorname{tg}x}{x}}_{\to 1}\right)}$$

$$\frac{1+2x+x^2-1}{x} = \frac{2x+x^2}{x} = 2+x$$

$$\approx \lim_{x\to 0}\frac{1+1}{2+x+1} = \frac{2}{3}$$

25) $$\lim_{x\to\infty}\left(\frac{1+|\sin x|}{x}\right)^x = \lim_{x\to\infty} e^{x\log\left(\frac{1+|\sin x|}{x}\right)}$$

$$= \lim_{x\to+\infty} e^{x\underbrace{\log(1+|\sin x|)\frac{|\sin x|}{|\sin x|}}_{\to 1}-x\log x}$$

$$\approx \lim_{x\to+\infty} e^{x|\sin x|-x\log x} = \lim_{x\to+\infty} e^{\underbrace{x}_{\to+\infty}(\underbrace{|\sin x|}_{\to[0,1]}-\underbrace{\log x}_{\to+\infty})}$$

$$e^{-\infty} = 0$$

WRONG because $|\sin x| \not\to 0 \quad x\to+\infty$

$$\lim_{x\to+\infty} e^{x\left(\underbrace{\log(1+\underbrace{|\sin x|}_{\to[0,1]})}_{\to[0,\log 2]}-\log x\right)} = e^{-\infty} = 0$$

33) $$\lim_{x\to 0}\frac{1}{1-\cos 2x}-\frac{1}{2\operatorname{tg}^2x} = \lim_{x\to 0}\frac{2\operatorname{tg}^2x+\cos 2x-1}{2(1-\cos 2x)\operatorname{tg}^2x}$$

$$= \lim_{x\to 0}\frac{x^2\left(2\underbrace{\frac{\operatorname{tg}^2x}{x^2}}_{\to 1}+4\underbrace{\frac{\cos 2x-1}{4x^2}}_{\to\frac{1}{2}}\right)}{2(1-\cos 2x)\underbrace{\operatorname{tg}^2x}_{\to 1}}$$

$$= \lim_{x\to 0}\frac{(2+4)}{2(1-\cos 2x)} = +\infty$$

$$= \lim_{x\to 0} \frac{\quad}{2(1-\underbrace{\cos 2x)}_{\to 0^+}} = +\infty$$

29) $$\lim_{x\to 1} \frac{\cos\left(\pi \frac{x}{2}\right)}{x-1} \underset{\substack{t=x-1\\ x=t+1\\ x\to 1,\ t\to 0}}{=} \lim_{t\to 0} \frac{\cos\left(\pi \frac{t+1}{2}\right)}{t}$$

$$\pi \frac{t+1}{2} \to \frac{\pi}{2} + \frac{t\pi}{2}$$

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$

$$= \lim_{t\to 0} \underbrace{\frac{\sin\left(-\frac{t\pi}{2}\right)}{t\left(-\pi/2\right)}}_{\to 1} \left(-\pi/2\right) = -\frac{\pi}{2}$$

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LIMITS

1) $\lim\limits_{n\to+\infty} \dfrac{n\sqrt{n}-\sqrt{n^2+4}-3}{n\sqrt{n}\,\mathrm{arctg}(n)+n\tanh(n)}$

2) $\lim\limits_{n\to+\infty} \sqrt{n^4-n^2}-n^2$

3) $\lim\limits_{n\to5} \dfrac{n-\sqrt{n^2+n-5}}{1-\sqrt{n-4}}$

4) $\lim\limits_{n\to\infty} \sqrt[3]{n^3-n}-n$

5) $\lim\limits_{n\to0} \dfrac{n-\sin(3n)}{n-\sin(2n)}$

6) $\lim\limits_{n\to\alpha} \dfrac{\sin n-\sin\alpha}{n-\alpha}$

7) $\lim\limits_{n\to0} \dfrac{\sin(n^2-n)}{n}$

8) $\lim\limits_{n\to0} \dfrac{1+\sin n-\cos n}{1-\sin n-\cos n}$

9) $\lim\limits_{n\to0} \dfrac{1-\cos^3 n}{\sin^2 n}$

10) $\lim\limits_{n\to1} \dfrac{\sin(\pi n^2)}{n-1}$

11) $\lim\limits_{n\to0} \dfrac{\cos(\alpha n)-\cos(\beta n)}{n^2}$

12) $\lim\limits_{n\to0} \dfrac{\cos(n)-\cos(2n)}{1-\cos n}$

13) $\lim\limits_{n\to0} \dfrac{\log(\cos n)}{\sqrt{1+n^2}-1}$

14) $\lim\limits_{n\to0} \dfrac{\log(2-\cos(2n))}{n^2}$

15) $\lim\limits_{n\to1^+} \dfrac{n^n-1}{(\log n-n+1)^2}$

16) $\lim\limits_{n\to-1} \dfrac{n+1}{\sqrt[4]{n+17}-2}$

17) $\lim\limits_{n\to0} \dfrac{\arccos n-\pi/2}{n}$

18) $\lim\limits_{n\to\infty} \dfrac{\lfloor 3n+1\rfloor}{\sqrt{n^2+1}}$

21) $\lim\limits_{x\to0} \dfrac{x^2+x\sin x}{1-\cos x}$

22) $\lim\limits_{x\to0} \dfrac{e^x-1+\sin x}{(1+x)^2-1+\mathrm{tg}\,x}$

23) $\lim\limits_{x\to0} \dfrac{e^{2x}-\sqrt{1+x}}{\mathrm{tg}\,x}$

24) $\lim\limits_{x\to\infty} x\left(a^{1/x}-1\right)$

25) $\lim\limits_{x\to\infty} \left(\dfrac{1+|\sin x|}{x}\right)^x$

26) $\lim\limits_{x\to0} \dfrac{x\cos x+x^2-\sin(2x)}{x+\sqrt{|x|}}$

27) $\lim\limits_{x\to-\infty} \dfrac{|x|^{3/2}+e^x}{2\sqrt{|x^3+3|}}$

28) $\lim\limits_{x\to\pi} \dfrac{e^{x-\pi}-\sin(3x)-1}{x-\pi}$

29) $\lim\limits_{x\to1} \dfrac{\cos(\pi x/2)}{x-1}$

30) $\lim\limits_{x\to+\infty} \left(1+\frac{1}{x}\right)^{\log(2\sinh x)}$

31) $\lim\limits_{x\to+\infty} (1+\sinh x)^{1/x}$

32) $\lim\limits_{x\to0} \dfrac{e^{2x}-\cos x}{\sin x}$

33) $\lim\limits_{x\to0} \dfrac{1}{1-\cos 2x}-\dfrac{1}{2\,\mathrm{tg}^2 x}$

34) $\lim\limits_{x\to+\infty} \left(\dfrac{x^4+2x^3+x}{x^4}\right)^{3x}$

35) $\lim\limits_{x\to1^+} \left[\sin(\ln x)+\cos(3\ln x)\right]^{\frac{1}{x-1}}$

36) $\lim\limits_{x\to0} \cos x^{\frac{1}{x}}$

37) $\lim\limits_{x\to1} x^{\frac{x}{1-x}}$

38) $\lim\limits_{x\to\infty} \left(e^{-2x}+\tanh x\right)^{e^{2x}}$

lim $2\sin(x-1)\left(e^{x^2-1}-1\right)$

$$\lim_{n\to\infty} \frac{\quad}{\sqrt{n^2+1}}$$

19) $$\lim_{n\to+\infty} \binom{n}{2} \Big/ \binom{n}{3}$$

20) $$\lim_{n\to+\infty} \frac{(n+2)!-n!}{(n+1)!\,(2n+1)}$$

38) $$\lim_{x\to\infty} (e \quad + \tanh x)$$

39) $$\lim_{x\to1} \frac{2\sin(x-1)\left(e^{x^2-1}-1\right)}{x^3-x^2-x+1}$$

40) $$\lim_{x\to0} \frac{\log(2-\cos(2x))}{x^2}$$

1) $$\lim_{n\to+\infty} \frac{\sqrt{n^3+9n^2}-\sqrt{n^4+1}}{n^2+2}$$

2) $$\lim_{n\to+\infty} \frac{\sqrt{n^3+9n^2}-\sqrt{n^4+1}}{n^2+2n}$$

3) $$\lim_{x\to+\infty} (x+\sqrt{x})\sin\left(\frac{5}{x}\right)$$

4) $$\lim_{x\to0^+} 4x^4\log(x^5+x^2)$$

5) $$\lim_{x\to0} \frac{1-\cos x^2}{x(1-\sin x)+x^3\sin^2 x}$$

6) $$\lim_{x\to0} \frac{\sin x^3}{x(1-\cos x)+x^4}$$

7) $$\lim_{x\to0} \frac{3}{x^3\sin\left(\frac{1}{x^3}\right)}\left(1-\frac{5}{x}\right)^x$$

8) $$\lim_{x\to0} \frac{3\operatorname{arctg} x+(1-\cos 2x)\sin^2 x}{27x^4+5\sin x}$$

9) $$\lim_{x\to0} \frac{e^{1-\cos x}-e^{\sin x}}{\log(1+3x)}$$

10) $$\lim_{x\to\pi^-} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{1-\cos^2 x}$$

11) $$\lim_{x\to+\infty} \frac{2\log\left(\cos\frac{1}{x}\right)+\alpha\left(\sin\frac{1}{x}\right)^2}{\left(\frac{1}{x}\right)^e}, \qquad \alpha\in\mathbb{R}$$

12) $$\lim_{x\to0} \frac{1-e^{-\alpha x^2}+x^5\sin\left(\frac{1}{x}\right)}{x^2}, \qquad \alpha\in\mathbb{R}$$