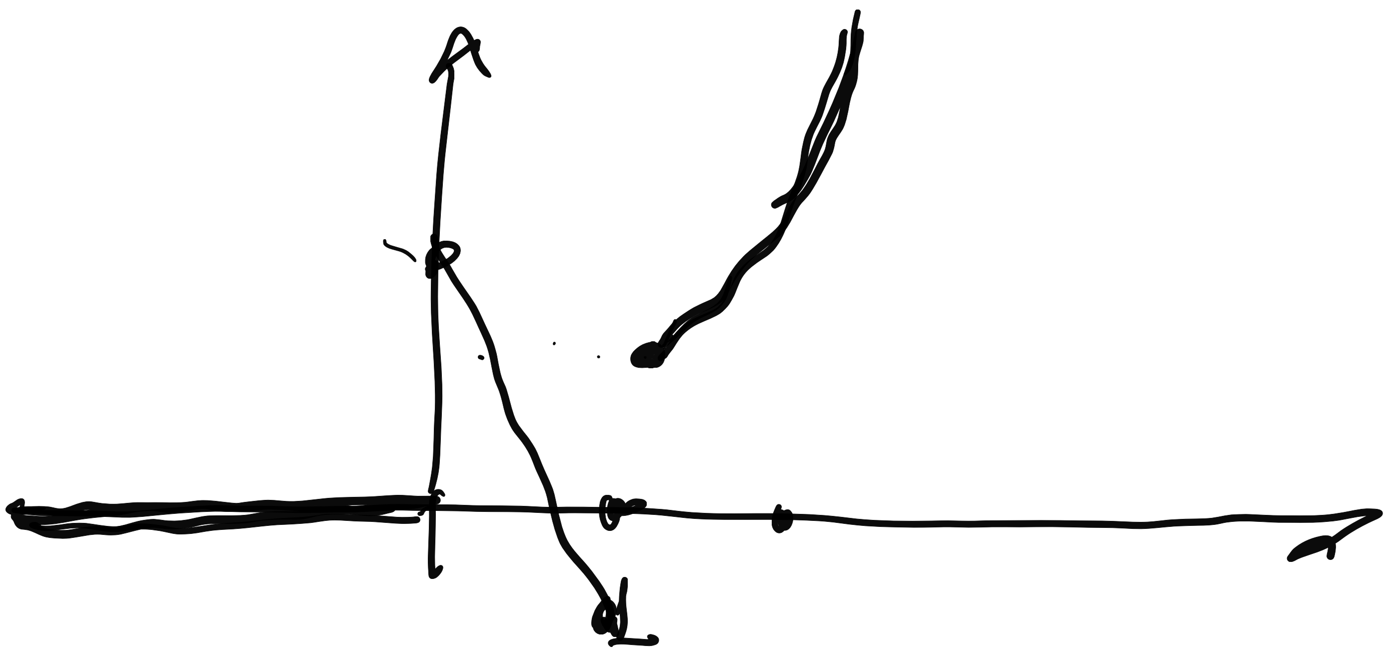
Example. $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) := \begin{cases} 0 & x < 0 \\ ax + b & 0 \leq x \leq 1 \\ x^2 & x > 1 \end{cases}$$



Determine a and b so that f is continuous at $x = 1$ and $x = 0$

$$\lim_{x \to 0-} f(x) \overset{?}{=} \lim_{x \to 0+} f(x) \overset{?}{=} b$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} 0 = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ax + b = b$$

$$\Longrightarrow 0 = b$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ax =$$

$$\parallel$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1$$

$$\Longrightarrow \lim_{x \to 1^-} ax = 1$$

$$\Updownarrow$$

$$a = 1$$

f is continuous on $\mathbb{R}$

if and only if

(iff)

$a = 1 \quad b = 0$

Let $f: D \longrightarrow \mathbb{R}$ $\xi \in Acc(D)$

$\xi, l \in \mathbb{R}$ $\lim_{x \to \xi} f(x) = l$ $(*)$

$$\Updownarrow$$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $(**)$

$l - \varepsilon \leq f(x) \leq l + \varepsilon \quad \forall x \in]\xi - \delta, \xi + \delta[$

Prove "$\Downarrow$"

By contradiction

$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists \bar{x} \in]\xi - \delta, \xi + \delta[\cap D$

s.t. $f(\bar{x}) \notin [l - \varepsilon, l + \varepsilon]$

Choose $\delta = \frac{1}{n}$

$\exists \ \bar{x}_n \in]\xi - \frac{1}{n}, \xi + \frac{1}{n}[: f(\bar{x}_n) \notin [l - \varepsilon, l + \varepsilon]$

Consider the sequence $(\bar{x}_n)$. Observe that

$\xi - \frac{1}{n} < \bar{x}_n < \xi + \frac{1}{n}$ (and $x_n \in D$)

$\Longrightarrow \quad \overline{x}_n \longrightarrow \xi$

<u>Now</u> $\quad \underline{f(\overline{x}_n) \notin [l-\varepsilon, l+\varepsilon]}$

$f(\overline{x}_n) = \overline{y}_n$

$$\Downarrow$$

$$\lim_{n\to\infty} f(\overline{x}_n) \neq l$$

this contradicts the hypothesis

So "$\Downarrow$" is proved

Prove "$\Uparrow$", by exercise

Proposition: Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$

$\xi \in Acc(D)$.

If $\lim_{x \to \xi} f(x)$ exists then it is unique

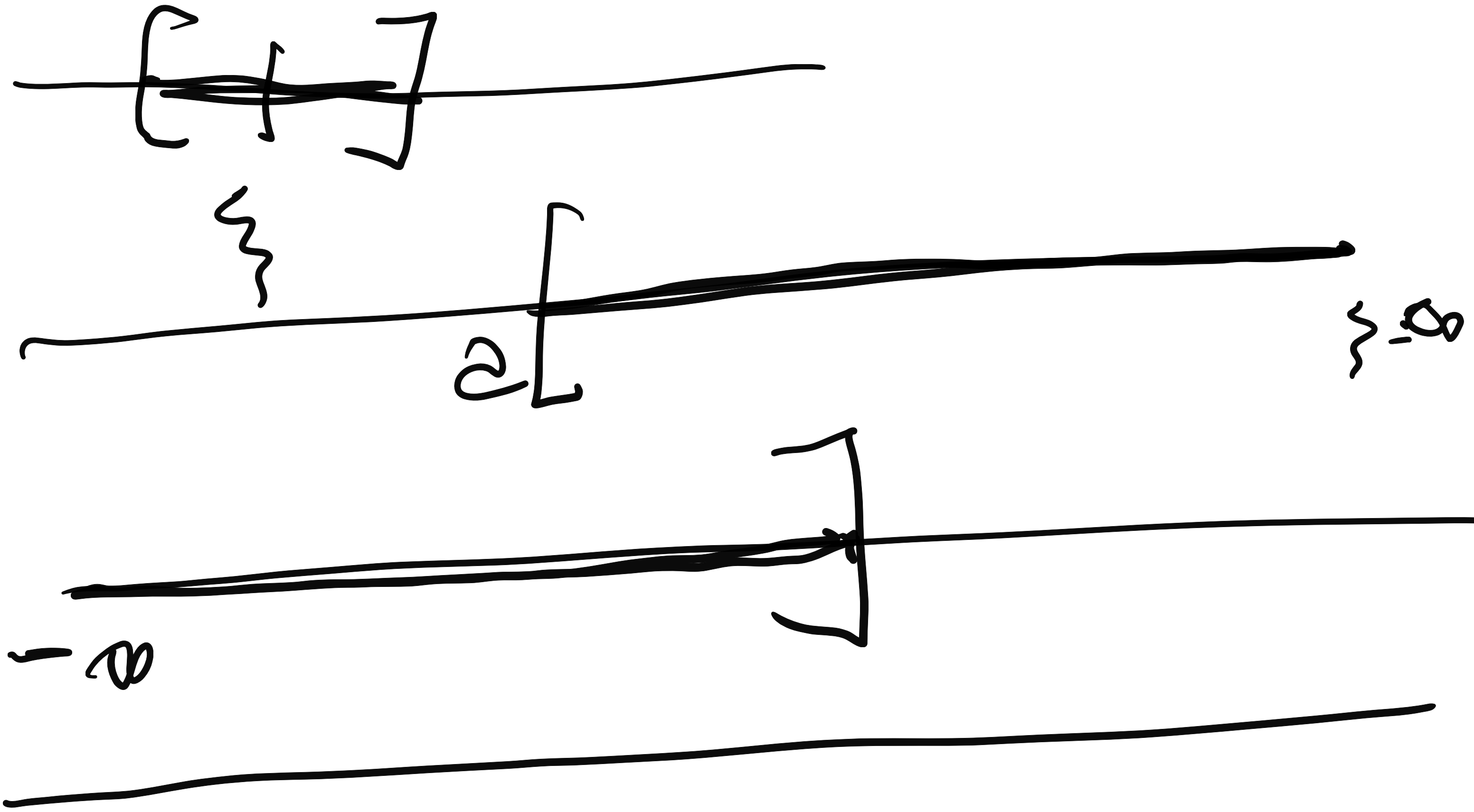
Definition $\xi \in \mathbb{R} \cup \{+\infty, -\infty\}$

A neighbourhood of ξ is:

- $[\xi - r, \xi + r]$ for some $r >$ if $\xi \in \mathbb{R}$
- if $\xi = +\infty$: $[a, +\infty[$ for some $a \in \mathbb{R}$

- if $\xi = -\infty$

$]-\infty\ b]$ for some $b \in \mathbb{R}$



Theorem: $f: D \to \mathbb{R}$ $\xi \in Acc(D)$

Assume $\exists \lim_{x \to \xi} := l \in \mathbb{R} \cup \{\pm\infty\}$

Then

1) if $l > 0$ there exists a neighbourhood I of ξ such that $f(x) > 0$ $\forall x \in I$

2) if $f(x) \geq 0$ $\forall x \in I$ and $\lim f(x) = l$ $\Rightarrow l \geq 0$

I neighb. of ξ

Prove 1: By contradiction

For every neighbourhood I of ξ

there exists $x \in I$ s.t.

$f(x) \leq 0$

take $I = I_n = [\xi - \frac{1}{n}, \xi + \frac{1}{n}]$

$\exists x_n \in [\xi - \frac{1}{n}, \xi + \frac{1}{n}]$ s.t.

$\boxed{f(x_n) \leq 0}$ $\forall n \in \mathbb{N}$

$x_n \longrightarrow \xi$

$f(x_n) \longrightarrow l$

$\Longrightarrow l \leq 0$ in contradiction

with $l > 0$.

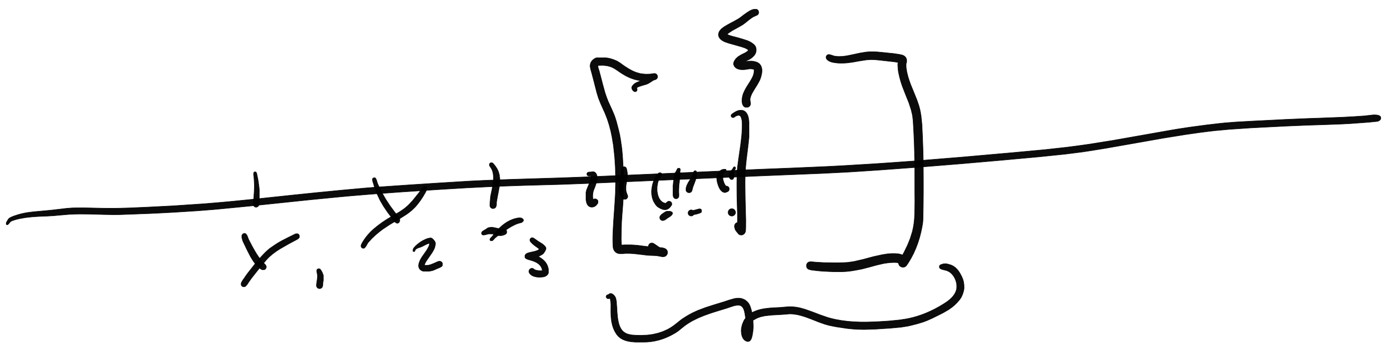
Theorem: Two policeman

$$\boxed{f(x) \le g(x) \le h(x)}$$

for $x \in I$ neighb. of ξ

$\lim_{x \to \xi} f(x) = \ell$ $\quad$ $\lim_{x \to \xi} h(x) = \ell$

$$\exists \; \boxed{\lim_{x \to \xi} g(x) = \ell}$$



$(x_n) \to \xi$ $\quad \exists \bar{N}$ s.t.

$n > N$ $\quad f(x_n) \in I$

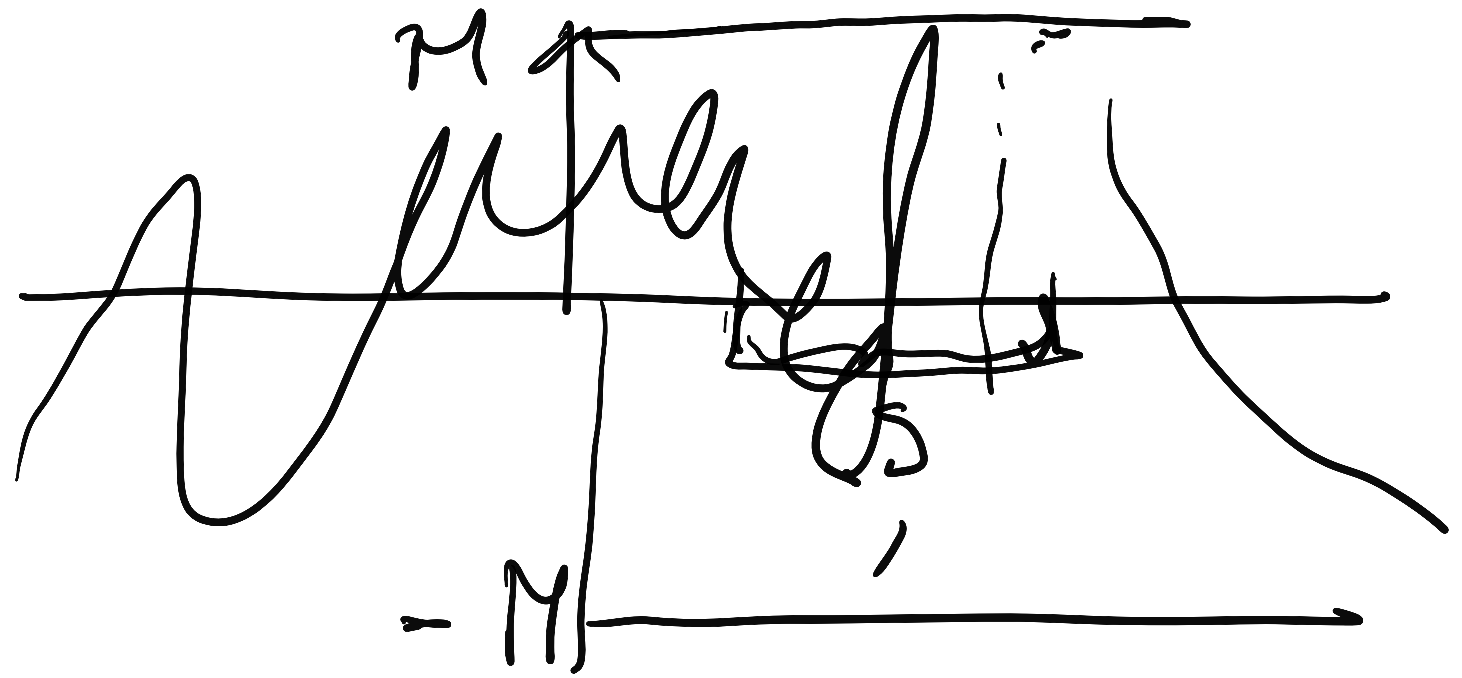
$$\ell \leftarrow f(x_n) \le g(x_n) \le h(x_n) \to \ell$$

$\Longrightarrow \lim_{n\to+\infty} g(x_n) = \ell$

for ~~pdilemer~~ Th

Definition: We say that a function $f: D \to \mathbb{R}$ is BOUNDED on $S \subset D$ if

$$\exists M \geq 0 : |f(x)| \leq M \quad \forall x \in S.$$

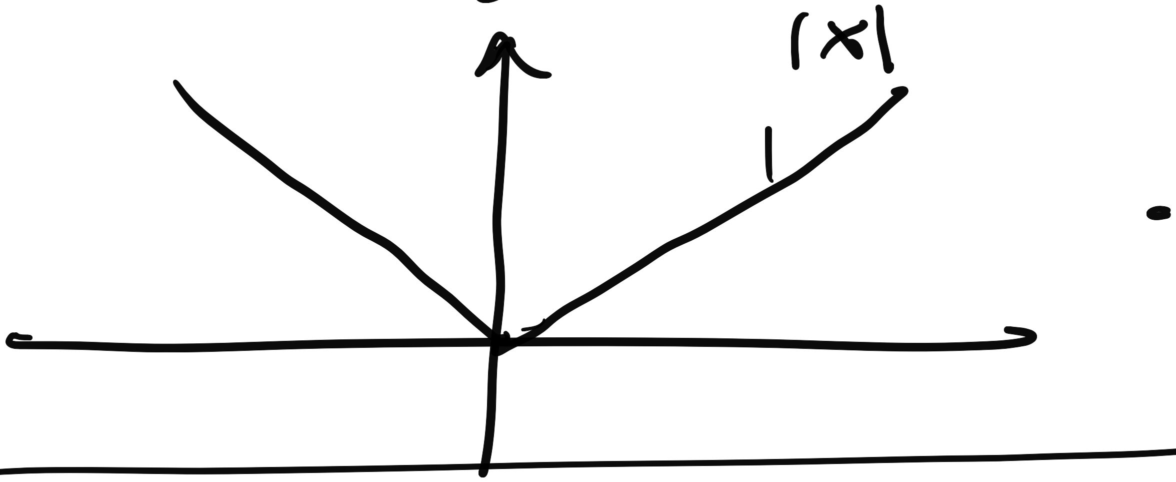


Definition $f: D \to \mathbb{R}$, $\xi \in Acc(D)$

We say that f is **infinitesimal at ξ** if $\exists \lim_{x \to \xi} f(x) = 0$

Examples: $f: x \longmapsto |x|$ is infinitesimal at 0

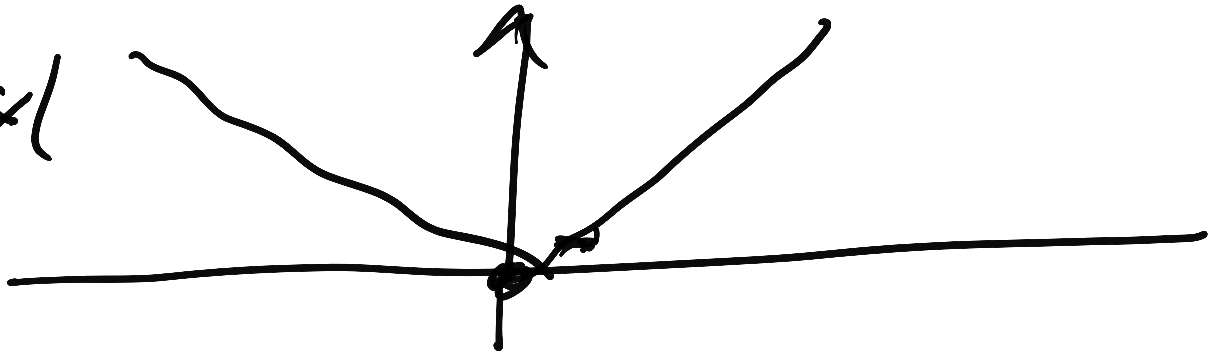
$$\lim_{x \to 0} |x| = 0$$



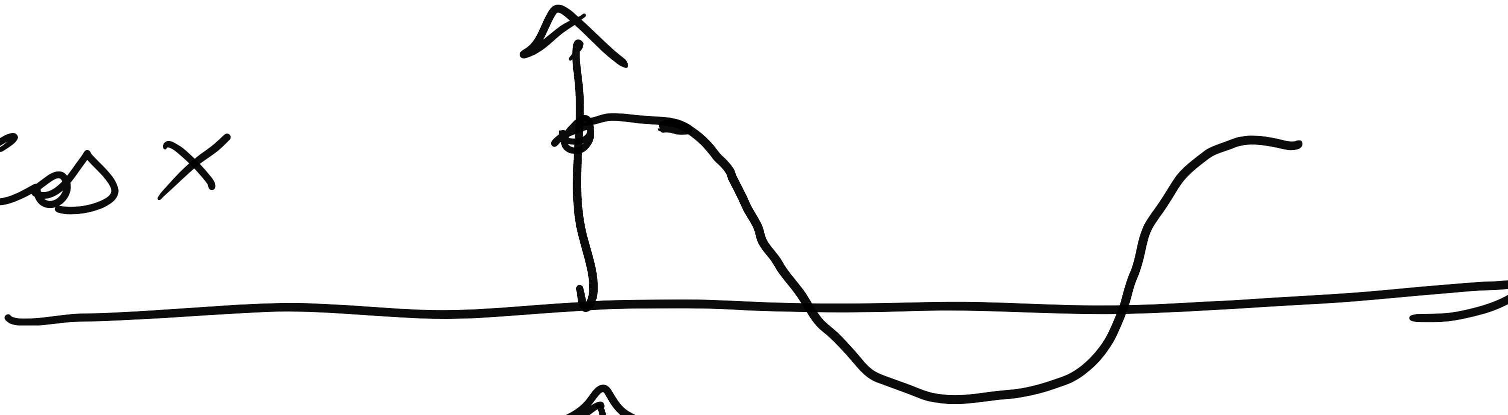
$f: x \longmapsto |x| \cdot \cos(x)$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| \cos(x) = 0$$

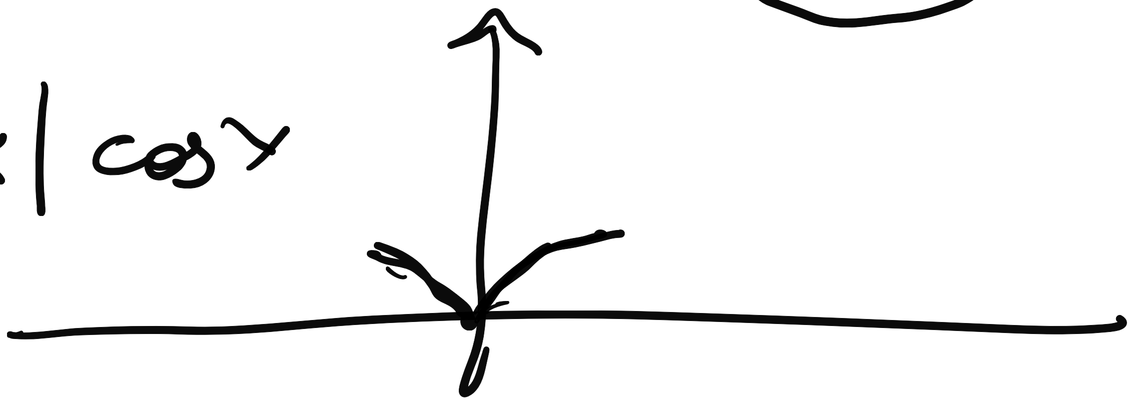
$|x|$



$\cos x$

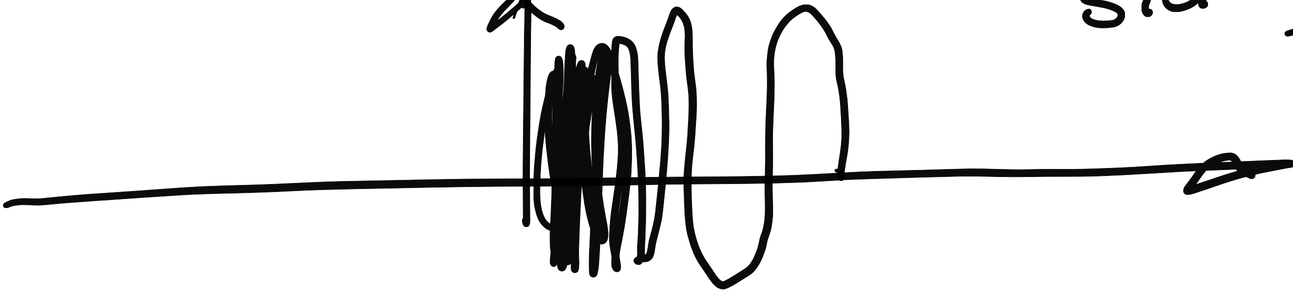


$|x| \cos x$



$f(x) = x \sin \frac{1}{x}$

$\sin \frac{1}{x}$



$\lim\limits_{x\to 0+} \sin\frac{1}{x}$ doesn't exist. $f = \sin\frac{1}{x}$

Indeed: $x_n = \dfrac{1}{\frac{\pi}{2}+2n\pi} \longrightarrow 0$

$$\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty}\sin\left(\frac{1}{x_n}\right) = \lim_{n\to+\infty}\sin\underbrace{\left(\frac{\pi}{2}+2n\pi\right)}_{\sin\frac{\pi}{2}=\boxed{1}} = 1$$

$\hat{x}_n = \dfrac{1}{2n\pi}$ $\quad f(\hat{x}_n) = \sin\frac{1}{\hat{x}_n} = 0$

$$\lim_{n\to+\infty} f(\hat{x}_n) = \boxed{0}$$

$$\Longrightarrow \boxed{\lim_{x\to 0+}\sin\frac{1}{x} \text{ does not exist}}$$

PROVE: $\lim\limits_{x\to 0-} \sin\left(\frac{1}{x}\right)$

The function $\sin\frac{1}{x}$ is BOUNDED, $\left|\sin\frac{1}{x}\right| \le 1$ on $\mathbb{R}\setminus\{0\}$

Proposition: $f, g : D \to \mathbb{R}$, $\xi \in Acc(D)$

f bounded on D and $\lim_{x\to\xi} g(x) = 0$

$$\Longrightarrow \exists \lim_{x\to\xi} f(x)\cdot g(x) = 0$$

Proof

$$0 \leq |\overbrace{f(x)g(x)}| =$$

$$= |\underbrace{f(x)}_{M}|\,|g(x)| \leq \underbrace{M|g(x)|}_{\to 0}$$

$$g(x) \to 0$$

Apply two policemen Th.

$$\lim_{x\to\xi} |f(x)\cdot g(x)| = 0$$

$$\Downarrow$$

$$\lim_{x\to\xi} f(x)\cdot g(x) = 0$$

q.e.d.

Apply to

$$\lim_{x\to 0} \underbrace{\sin(x)}_{\to 0} \cdot \underbrace{\sin\frac{1}{x}}_{\leq 1}$$

$0 \leq \sin(x) \leq x \quad x>0$ ... $x<0$

$$\Longrightarrow \lim_{x\to 0} \sin x = 0$$

$$\Longrightarrow \lim_{x\to 0} \sin(x) \sin\left(\frac{1}{x}\right) = 0$$

Indeterminate cases:

1) $\lim\limits_{x\to 0} \dfrac{g(x)}{f(x)}$ —— $\lim g(x) = 0$, $\lim f(x) = 0$

$g(x) = x^3$

$f(x) = \sin x$

$$\lim_{x\to 0^+} \frac{x^3}{\sin x} = \lim_{x\to 0} \underbrace{\frac{x}{\sin x}}_{\to \boxed{1}} \cdot \underbrace{x^2}_{\to 0} = \underline{0}$$

$g(x) = x^3$

$f(x) = (\sin x)^5$

$$\lim \frac{x^3}{(\sin x)^5} = \underbrace{\frac{x^5}{(\sin x)^5}}_{\to 1} \cdot \frac{1}{x^2} =$$

But

$$\frac{x^5}{(\sin x)^5} = \left(\frac{x}{\sin x}\right)^5 \longrightarrow 1$$

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty \Longrightarrow \lim \frac{x^3}{(\sin x)^5} = +\infty$$

$$x_n = \frac{1}{1000\, n}$$

$$\frac{1}{x_n^2} = 1000 \cdot 000\, n^2 \longrightarrow +\infty$$