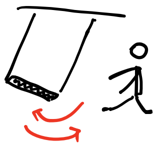
Lesson 14 - 27/10/2022

- The mechanism of the mechanical clock is the same of the swing! (see video...)



- The "simplest" Lotka-Volterra model is not structurally stable (it is sufficient to consider the modified L-V model, for $\varepsilon > 0$ very small).
More realistic models take into account: the carrying capacity of the two populations and the type of response of the predator ... we obtain a LIMIT CYCLE (see video ...)

- One-dimensional maps.
- The logistic map $f : [0,1] \circlearrowleft$, $f(x) = 2x(1-x)$.
- $\begin{cases} \dot{x} = 2x(1+y) - \sin(x+y) \\ \dot{y} = xy - y \end{cases}$ $(*)$

· Det. equilibria (recall that $2t = \sin t \iff t = 0$)
· Linearize around the equilibria
· Draw the phase-portraits of the Linearized systems
· what about the stability of equilibria in $(*)$? (original, non-linear system)
· Det. hyperbolic and elliptic equilibria of $(*)$. In the hyp. case, det. the dim. of the stable / unstable mnfs.
- Draw the phase-portrait for $\ddot{x} = x^3 + x^2$. $\to$ At home, we will solve it on Wedn.

$V(x) = -\frac{1}{4}x^4 - \frac{1}{3}x^3$

One dim-maps.

Mechanical clock : $f(v) = av$ (without external force)

$f(v) = av + b$ (with const. external force)

- v_0, $\underbrace{av_0}_{v_1}$, $av_1 = a^2 v_0$ ....

$v_k = a^k v_0$

- v_0, $\underbrace{av_0 + b}_{v_1}$, $av_1 + b$ ...

$v_{k+1} = a v_k + b$

This is an example of discrete din. system, given by the iteration of a map.
(continuous map).
As in the continuous case (flows), we are interested on equilibria and stability of equilibria for discrete dynamical systems.
Discrete dynamical systems are important since:
- They are tools for analyzing differential eqs. (by discretization).
- They are model for natural phenomena (growth population, think also to the swing / mechanical clock...)
- They are simple examples of chaos.

—x—x—

$f \in C^\infty : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$.

$$\mathrm{Orb}(x) = \{ f^k(x), k \in \mathbb{Z} \} = \{ \dots f^{-2}(x), f^{-1}(x), \underset{f^0(x)}{\underset{\parallel}{x}}, f^1(x), f^2(x) \dots \}$$

gives a discrete dynamics by iteration:

$$\boxed{x_{n+1} = f(x_n)}$$

- Equilibria $\to$ Fixed points of f

Suppose $x^* \in \mathbb{R}$ is s.t. $f(x^*) = x^*$.

Then x^* is an equilibrium since :

$$\mathrm{Orb}^+(x^*) = \{ x^*, \underset{\underset{f(x^*)}{\shortparallel}}{x^*}, \underset{\underset{f^2(x^*)}{\shortparallel}}{x^*}, \dots \} = \{x^*\}$$

- <u>Stability of x^* ?</u>

We consider $x^* + \eta$ (η is small)

$$\underset{\underset{\text{Taylor}}{\downarrow}}{f(x^* + \eta) =} f(x^*) + f'(x^*)\eta + \mathcal{O}(\eta) \underset{\underset{\text{fixed point of } f}{\underset{x^* \text{ is a}}{\downarrow}}}{=}$$

$$= x^* + f'(x^*)\eta + \mathcal{O}(\eta)$$

that is

$$x^* + \eta \longmapsto x^* + f'(x^*)\eta + \mathcal{O}(\eta)$$

which means (neglecting $\mathcal{O}(\eta)$ - terms)

$$\eta \longmapsto \underset{\underset{\lambda}{\shortparallel}}{\boxed{f'(x^*)}}\eta$$

And so

$$\mathrm{Orb}^+(\eta) = \{ \eta, \lambda\eta, \lambda^2\eta, \lambda^3\eta \dots \}$$
$$= \{ \lambda^n \eta, n \in \mathbb{N} \cup \{0\} \}$$

<u>So</u>

- If $|f'(x^*)| = |\lambda| < 1$ then $\lim_{n \to +\infty} \lambda^n \eta = 0$ and x^* is called LINEARLY STABLE.

- If $|f'(x^*)| = |\lambda| > 1$ then x^* is called LINEARLY UNSTABLE

- The linearization cannot help us in the marginal case $|\lambda| = 1$.

EXAMPLE

$f(x) = x^2$

$x \longmapsto x^2 \longmapsto x^4 \longmapsto x^8 \ldots$

EQUILIBRIA ?

$f(x) = x^2 = x \iff x^2 - x = 0 \iff$

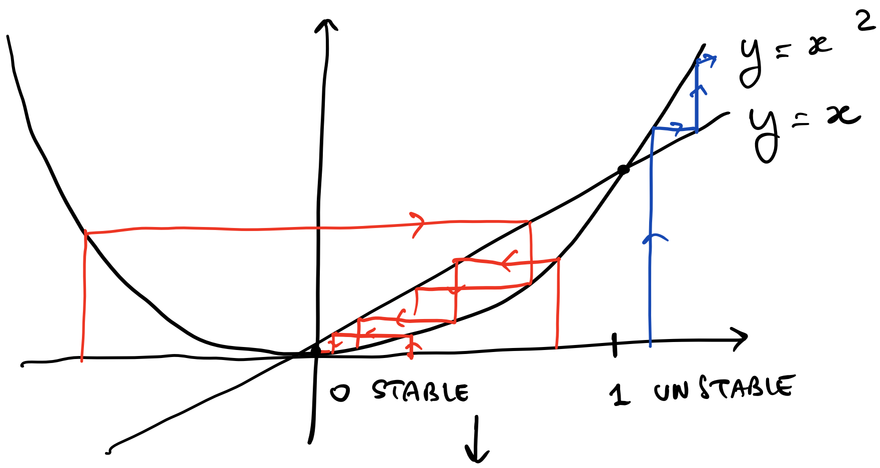
$x(x-1) = 0 \quad \begin{matrix} \nearrow & x = 0 \\ \searrow & x = 1 \end{matrix}$

THEIR STABILITY ?

$f'(x) = 2x$

$f'(0) = 0 \Rightarrow 0$ IS LINEARLY STABLE

$f'(1) = 2 \Rightarrow 1$ IS LINEARLY UNSTABLE.



↓ Cobweb Plot

THE LOGISTIC MAP

$f : [0,1] \to [0,1]$, $f(x) = rx(1-x)$, $r \geq 0$.

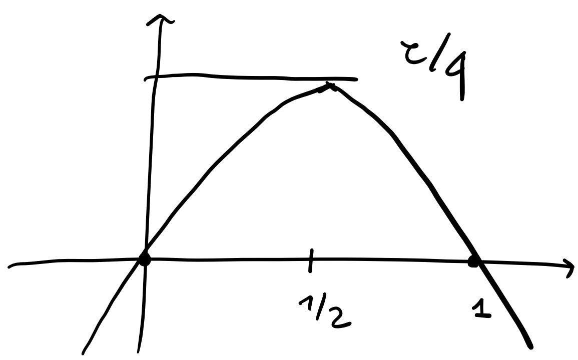
- In order that $f(x) = rx(1-x) \in [0,1]$ we need $\boxed{0 \leq r \leq 4}$. In fact

$$f'(x) = r(1-x) + rx(-1) =$$

$$= r - 2rx = 0 \iff x = 1/2$$

$$f(1/2) = \frac{r}{2}\left(\frac{1}{2}\right) = \frac{r}{4} \leq 1 \iff$$

$$r \leq 4$$



- Equilibria : $f(x) = x$

$\tau x(1-x) = x \iff \tau x - \tau x^2 = x$

$\iff \boxed{x = 0}$ OR $\tau - \tau x - 1 = 0$

$\iff \tau x = \tau - 1 \iff \boxed{x = \dfrac{\tau - 1}{\tau}}$

Divide by $\tau > 0$

<u>Then</u>

- The origin $x = 0$ is a fixed point (equilibrium) $\forall\ \tau \in [0, 4]$
- Whereas $x^* = 1 - \frac{1}{\tau}$ is in the range of allowable x only if $1 - \frac{1}{\tau} \geq 0$

that is $\frac{1}{\tau} \leq 1 \iff \tau \geq 1$.

- Moreover, the stability of these two equilibria depends on f'.

In particular $f'(x) = \tau - 2\tau x$

$f'(0) = \tau \rightarrow$ 0 is stable when $0 < \tau < 1$

$\searrow$ 0 is unstable when $\tau \in]1, 4[$.

Let now $\tau \geq 1$.

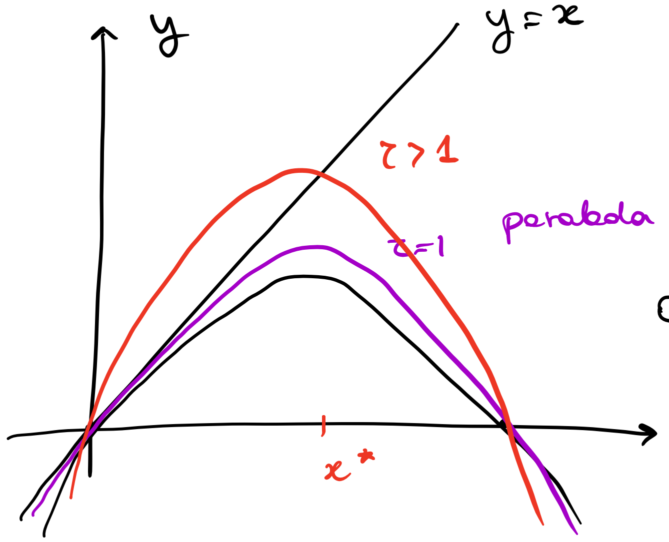
$$f'(x^*) = \tau - 2\cancel{\tau}\left(\frac{\tau - 1}{\cancel{\tau}}\right) = \tau - 2\tau + 2$$

$= -\tau + 2$

x^* is $\rightarrow$ stable if $|-\tau + 2| < 1 \iff -1 < -\tau + 2 < 1 \iff 1 < \tau < 3$

$\searrow$ unstable if $\tau > 3$ $(\tau \in]3, 4])$

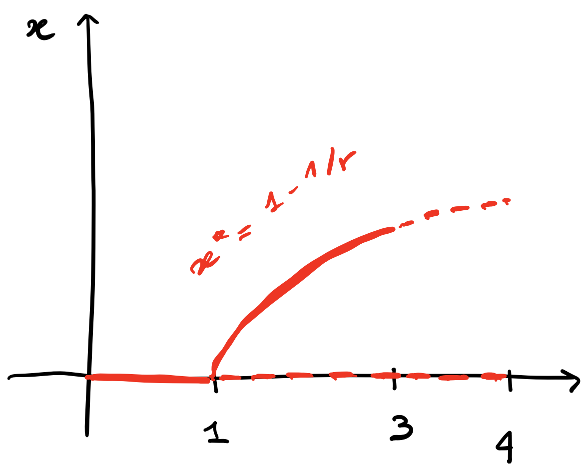
Remark



parabola tg to $y = x$ in 0.

$0 < r < 1$

Bifurcation diagram



EX

$$\begin{cases} \dot{x} = 2x(1+y) - \sin(x+y) \\ \dot{y} = xy - y \end{cases}$$

$xy - y = 0 \iff y(x-1) = 0 \rightarrow y = 0$ OR $x = 1$

$\underline{y=0}$: $2x - \sin x = 0 \iff x = 0$

$P_0 = (0,0)$

$\underline{x=1}$: $2(1+y) - \sin(1+y) = 0 \iff$

$1+y = 0 \iff y = -1$

$P_1 = (1,-1)$

$$Jx(x,y) = \begin{pmatrix} 2(1+y) - \cos(x+y) & 2x - \cos(x+y) \\ y & x-1 \end{pmatrix}$$

$$Jx(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = -y \end{cases}$$

Linear. around $(0,0)$

$$Jx(1,-1) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

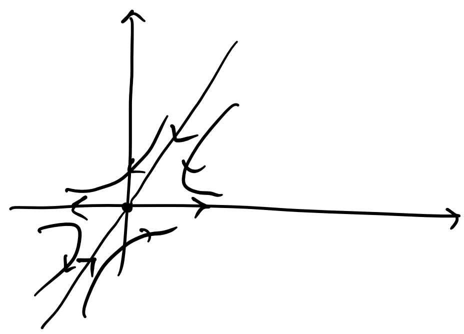
$$\begin{cases} \dot{x} = -(x-1) + (y+1) \\ \dot{y} = -(x-1) \end{cases}$$

Linear. around $(1,-1)$

$(0,0) \rightarrow \begin{cases} 1 & \text{with } v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ -1 & \text{with } v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{cases}$

EIGENV. FOR $JX(0,0)$

EIGENVECTORS



SADDLE

HYP. EQUILIBRIUM.
THE PHASE PORTRAIT AROUND $(0,0)$ IS a cont. deformation of this one. $\text{Dim } W^u(0,0) = \text{Dim } W^s(0,0) = 1$.

$(1,-1)$: Eigen. $JX(-1,1) \rightarrow \lambda_{1,2} = \dfrac{-1 \pm \sqrt{3}\, i}{2}$

Real part < 0

$(1,-1)$ is a stable spiral for the linearization.
By First Lyap. Theo. $(1,-1)$ IS ASYMPTOTICALLY STABLE for the original (non linear) system.

Hyperbolic : $\dim W^s(-1,1) = 2$
$\dim W^u(-1,1) = 0$.

- First partial exam : 11 NOVEMBER
  10:30
  Ex First 8

- 9 NOVEMBER
  Benettin talk !!