$\log_a n \ll n^\alpha \le b^n$

"higher order"

Was proven except

(*) $\underline{c_n \longrightarrow +\infty}$ $\beta > 0$

then

$$(c_n)^\beta \longrightarrow +\infty$$

We have to prove:

$\forall M \ \exists N \ n \geq N$

$$\boxed{(c_n)^\beta > M}$$

We know by (*) that $\forall K$

$\exists \Phi : \forall n \geq \Phi$

$C_n \geq K$

Choose $\boxed{K = \boxed{M^{\frac{1}{\beta}}}}$

$\boxed{\exists \tilde{\Phi} \quad \forall n \geq \tilde{\Phi}}$

$\boxed{C_n \geq M^{\frac{1}{\beta}}}$

take the power β of both (positive) terms

$\Longrightarrow C_n^{\beta} \geq M.$

Def We say (a_n) is infinitesimal if

$$\lim_{n \to \infty} a_n = 0$$

Example $\frac{1}{n^\alpha}$ infinitesimal $(\alpha>0)$

$\frac{1}{(n+6)^\beta}$ infinitesimal $(\beta>$

Def If (a_n) and (b_n) are infinitesimal we say that a_n is a higher order infinitesimal if

$$\boxed{\lim \frac{a_n}{b_n} = 0}$$

(provided $b_n \neq 0$ for n sufficiently large)

Example $a_n = \frac{1}{n^\alpha}$ $b_n = \frac{1}{(n+6)^\beta}$ $\alpha, \beta > 0$

If $\alpha > \beta$

$$\frac{\frac{1}{n^{\alpha}}}{\frac{1}{(n+6)^{\beta}}} = \frac{(n+6)^{\beta}}{n^{\alpha}} =$$

$$\frac{n^{\beta}\left(1+\frac{6}{n}\right)^{\beta}}{n^{\alpha}} = n^{\overbrace{\beta-\alpha}^{<0}} \underbrace{\left(1+\frac{6}{n}\right)^{\beta}}_{\substack{\downarrow \\ 1}}$$

$$\boxed{\frac{1}{n^{\alpha-\beta}}}\underbrace{\left(1+\frac{6}{n}\right)^{\beta}}_{1} \longrightarrow 0$$

$$\lim_{n\to\infty} \frac{\frac{1}{n^{\alpha}+n^{\gamma}+3n^{\mu}}}{\frac{1}{n^{\beta}+n^{\nu}+7n^{\omega}}}$$

$\alpha > \gamma$

$\alpha > \mu$

$\beta > \nu$

$\beta > \omega$

$$\cancel{\frac{1}{n^{\alpha}\left(1+n^{\gamma-\alpha}+3n^{\mu-\alpha}\right)}}$$

$$\to \frac{\dfrac{1}{n^{\alpha}\left(1+\underbrace{n^{\gamma-\alpha}}_{\to 0}+3\,n^{\omega-\alpha}\right)} \to 0}{\to \dfrac{1}{n^{\beta}\left(1+\underbrace{n^{\gamma-\beta}}_{\to 0}+7\,\underbrace{n^{\omega-\beta}}_{\to 0}\right)}}$$

$$= \frac{n^{\beta}\underbrace{(\ldots\ldots)}_{\to 1}}{n^{\alpha}\underbrace{(\ldots\ldots)}_{\to 1}} = n^{\beta-\alpha}\underbrace{(\ldots\ldots)}_{\to 1} =$$

$$= \begin{cases} 0 & \text{if } \beta < \alpha \\ 1 & \beta = \alpha \\ +\infty & \text{if } \beta > \alpha \end{cases}$$

Exercice:

$$\lim_{n\to\infty} \frac{\left(\frac{9}{10}\right)^n - n^3 \to -\infty}{\underline{\left(\frac{21}{10}\right)^n} - n^{100}\cdot 2^n}$$

Numerator: $\to -\infty$

Denominator: $\left(\frac{21}{10}\right)^n \left(1 - n^{100}\cdot \frac{2^n}{\left(\frac{21}{10}\right)^n}\right.$

$$= \left(\frac{21}{10}\right)^n \left(1 - n^{100} \left(2 \cdot \frac{10}{21}\right)^n\right)$$

$$= \underbrace{\left(\frac{21}{10}\right)^n}_{\to +\infty} \left(1 - \underbrace{\boxed{\frac{n^{100}}{\left(\frac{21}{20}\right)^n}}}_{\to 0}\right)$$

remember

$n^a << b^n$

$a > 0$

$b > 1$

$$= +\infty$$

Exercise $c > 0$

$$\lim_{n\to\infty} \frac{c^{2n} - n^2 c^{-3n} - (-1)^n n^{17}}{c^{\frac{5}{2}n} + n^3} = ⊛$$

Find the limit when $c > 0$.

Case $c > 1$:

Numerator: $c^{2n}\left(1 - \overset{\to 0}{n^2 c^{-5n}} - \overset{\to 0}{\boxed{\frac{(-1)^n n^{17}}{c^{2n}}}}\right)$

Denominator: $c^{\frac{5}{2}n}\left(1 + \frac{n^3}{c^{\frac{5}{2}n}}\right)$

$$⊛ \lim \frac{c^{2n}\Big(\underbrace{1+\underbrace{\cdots\cdots}_{\to 0}}\Big)\to 1}{c^{\frac{5}{2}n}\Big(1+\underbrace{\cdots\cdots}_{\to 0}\Big)\to 1} =$$

$$= \lim c^{-\frac{1}{2}n}\Big(\qquad \Big) \to 1$$

$$= \underbrace{\lim c^{-\frac{1}{2}n}} \cdot 1 = 0$$

Case $0 < c < 1$

Numerator

$$c^{2n} - n^2 c^{-3n} - (-1)^n n^{7} =$$

$$n^2 c^{-3n}\left(\underbrace{\frac{c^{5n}}{n^2}}_{\to 0} - 1 - \underbrace{(-1)^n n^{+5}\left(\frac{1}{c}\right)^{3n}\cdots}_{\to 0}\right)$$

$$= n^2 c^{-3n}\Big(-1 + \underbrace{\cdots\cdots}_{\to 0}\Big)$$

Denominator

$$c^{\frac{5}{2}n} + n^3 = n^3\left(\underbrace{\frac{c^{\frac{5}{2}n}}{n^3}}_{\to 0} + 1\right)$$

$$\lim_{n\to\cdot\cdot} \cdots = \frac{\cancel{n^2}\, c^{-3n} \left(-1 + \underbrace{\cdots}\right)^{\to 0}}{n^3 \left(1 + \overset{0}{\underbrace{\cdots}}\right)}$$

$$= \boxed{\frac{c^{-3n}}{n}} \left(\quad \right) \to -1$$

$$\frac{\left(\frac{1}{c}\right)^{3n}}{n} \underset{\to -1}{\left(\quad \right)} \to -\infty$$

$c = \underline{1}$

$$\lim \frac{1^{2n} - n^2 1^{-3n} - (-1)^n n^{17}}{1^{\frac{5}{2}n} + n^3} =$$

$$\lim_{n\to\infty} \frac{1 - n^2 - (-1)^n n^{17}}{1 + n^3}$$

$$\lim_{n\to\infty} \frac{n^{17}\left(\frac{1}{n^{17}}^{\to 0} - \frac{1}{n^{15}}^{\to 0} - (-1)^n\right)}{n^3\left(\underset{\to 0}{\frac{1}{n^3}} + 1\right)} \to$$

$$= \lim_{n \to \infty} \underbrace{\boxed{n^{34}}}_{\to +\infty} \cdot \frac{\left(\overset{\to 0}{\cdots} - (-1)^n\right)}{\left(\overset{\to 0}{\cdots} + 1\right)}$$

$n = 2k$ the subsequence

$$\lim_{k \to 0} \underbrace{(2k)^{14}}_{+\infty} \frac{\left(\overset{\to 0}{\cdots} - \overset{=1}{(-1)^{2k}}\right)}{\left(\overset{\to 0}{\cdots} + 1\right)} \to \boxed{-\infty}$$

$n = 2k+1$

$$(2k+1)^{14} \frac{\left(\overset{\to 0}{\cdots} - \underset{\text{nl}}{(-1)^{2k+1}}\right)^{1}}{\left(\overset{\to 0}{\cdots} + 1\right)} =$$

$$= \boxed{+\infty}$$

$a_n \to 1^+$ $\qquad b_n \to +\infty$

$\lim a_n^{b_n} =$ — finite (example $(1+\frac{1}{n})$)

$a_n = 1 + \frac{1}{n} \to 1$

$b_n = n \to +\infty$

$$\lim \left(1+\frac{1}{n}\right)^n = e$$

$$\lim \left(1+\frac{1}{n}\right)^{\boxed{2n}} =$$

$$= \lim \Bigg(\underbrace{\left(1-\frac{1}{n}\right)^n}_{\to e}\Bigg)^2 = e^2$$

Exercise

$a_n \to l_1$

$(a_n)^2 \to l_1^2$

$$\lim \left(1+\frac{1}{n}\right)^{n^2} =$$

$$\lim \Bigg(\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}\Bigg)^n = +\infty$$

$a_n \to l_1 > 1$

$b_n \to +\infty$

$a_n^{b_n} \to +\infty$

$$\lim a_n^{b_n} \qquad a_n \to 1 \qquad b_n \to +\infty$$

$$a_n^{b_n} = e^{\log(a_n^{b_n})} = \boxed{e^{b_n \log a_n}} \quad \begin{cases} \to e^{l} \\ \to 1 \\ \to \infty \end{cases}$$

$$b_n \cdot \log a_n \quad \begin{cases} \to l \\ \to 0 \\ \to \infty \end{cases}$$

$$\infty \cdot 0$$

$$\left(1+\frac{1}{n}\right)^{n^2} =$$

$$e^{\log\left(1+\frac{1}{n}\right)^{n^2}} = e^{n^2 \log\left(1+\frac{1}{n}\right)}$$

$$n^2 \underbrace{\log\left(1+\frac{1}{n}\right)}_{\frac{1}{n}} = n^2 \overbrace{\boxed{\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}}}}^{\to 1} \frac{1}{n} =$$

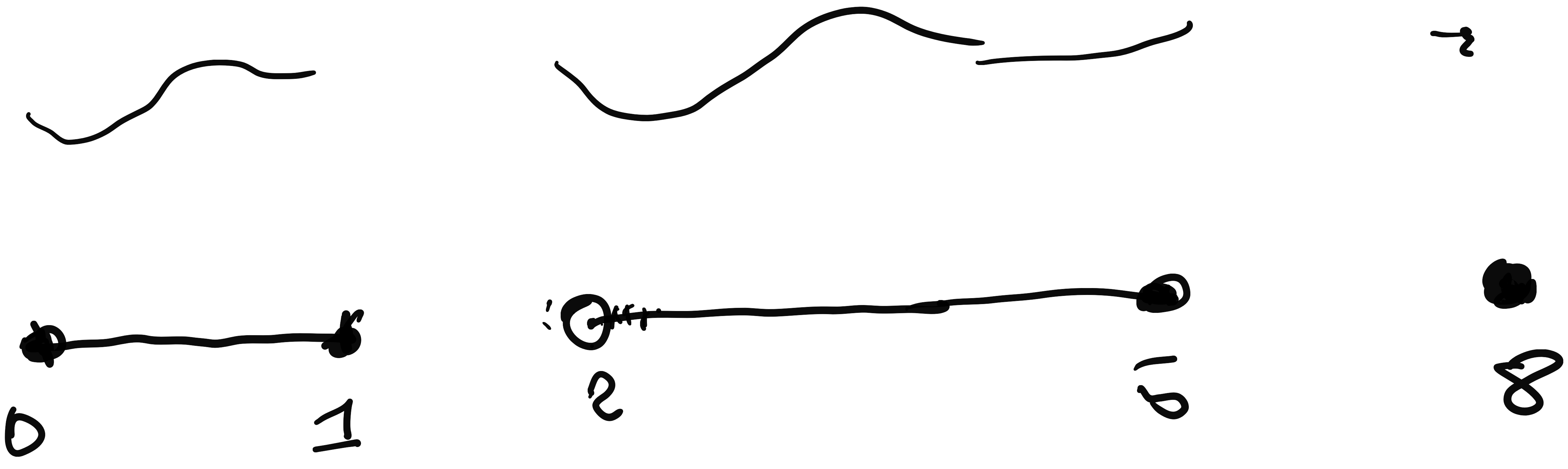
$$n \cdot \left(\quad \right)^{\to 1} \longrightarrow +\infty$$

$$\lim \left(1+\frac{5}{n}\right)^{\sqrt{n}} \longrightarrow 1$$

$$\underbrace{\left(\left(1+\frac{5}{n}\right)^{n}\right)^{\frac{1}{\sqrt{n}}}}$$

$$\left(\overset{\downarrow}{e^{\frac{1}{n}}}\right)^{\frac{1}{\sqrt{n}}} \longrightarrow 1$$

$$\begin{aligned} f: E &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) \end{aligned}$$



$$E = [0,1] \cup \left]2\ \ 5\right] \cup \{8\}$$