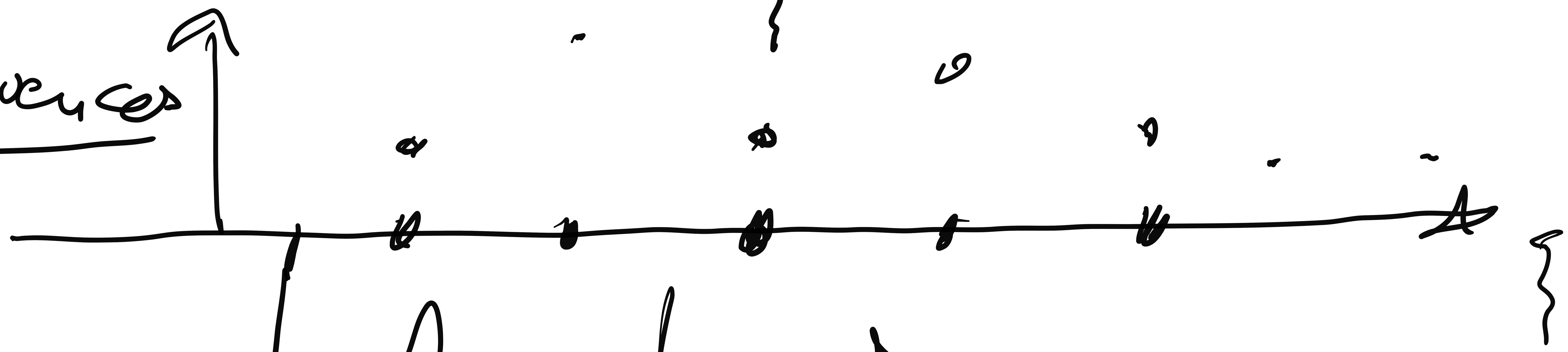
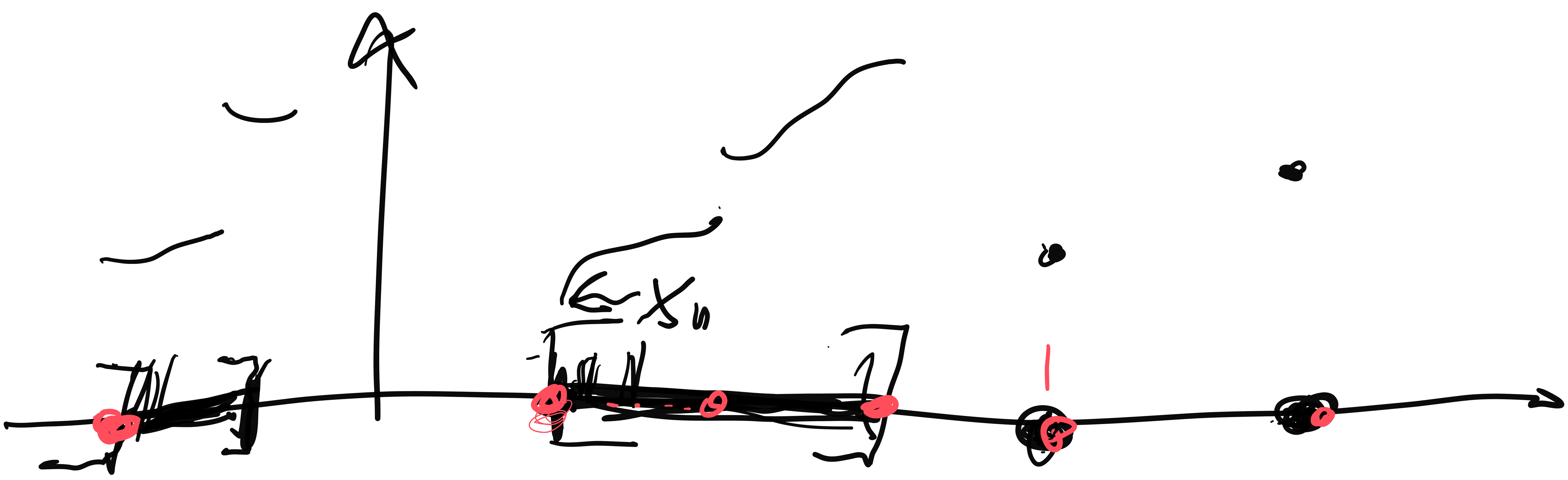
Goal: give a meaning to $\lim\limits_{x \to \xi} f(x)$

Sequences



General functions

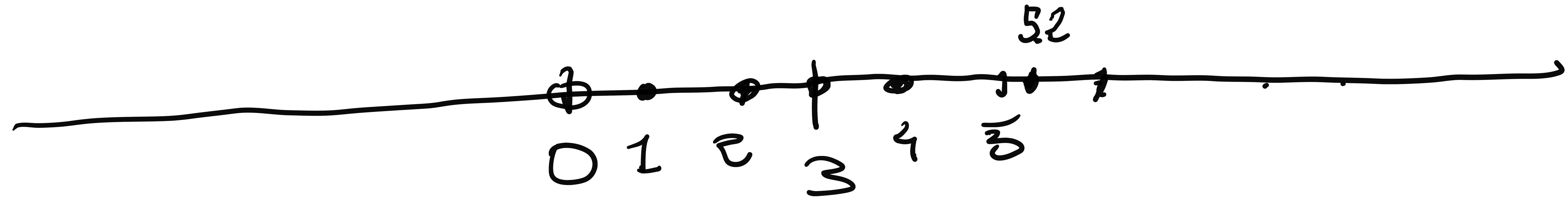


$f: D \longrightarrow \mathbb{R}$

Definition We say that $\xi \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is an accumulation point for a set $D \subseteq \mathbb{R}$ if there exists

(x_n), $x_n \in D \setminus \{\xi\}$ s.t. $x_n \longrightarrow \xi$

Example: $D = \,]0,3[\, \cup \mathbb{N}$



Find the accumulation points;

$\xi \in [0,3]$ are acc. pt.

$+\infty$ is acc. pt.

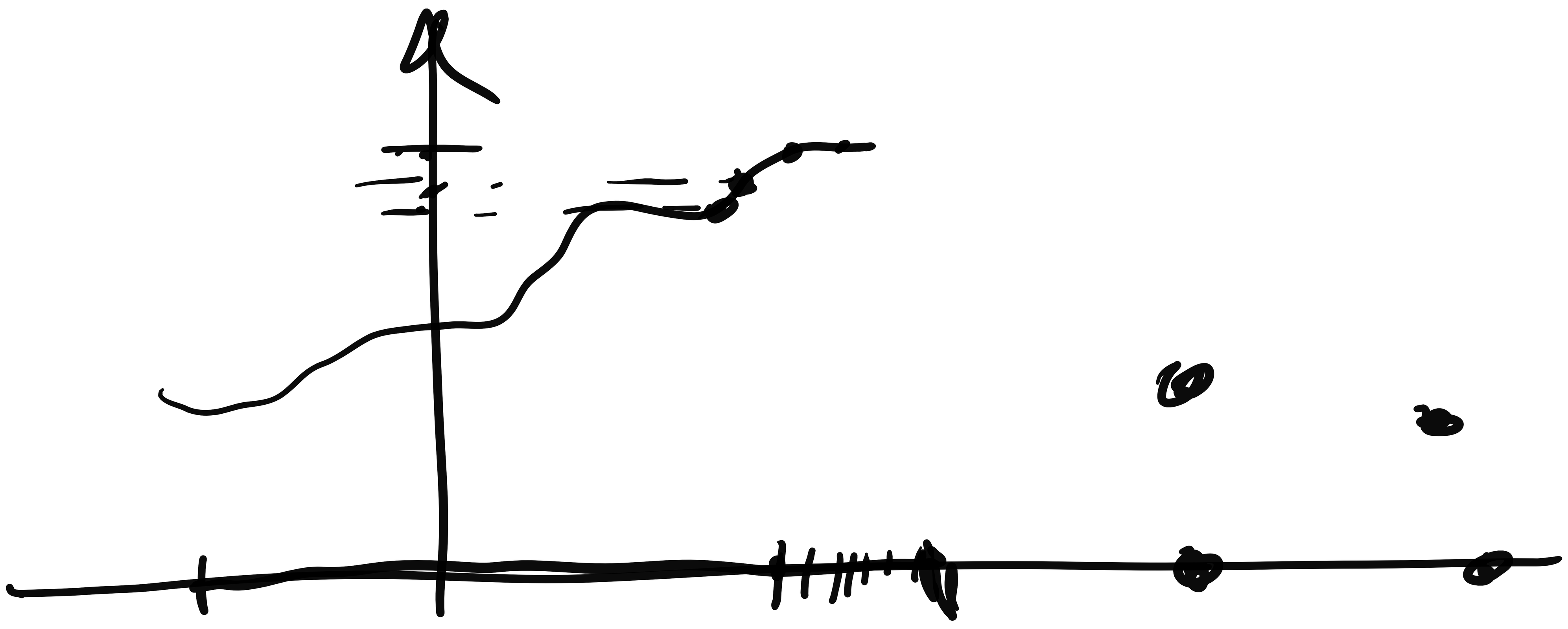
$Acc(D) = [0,3] \cup \{+\infty\}$

Set of acc. points of D ~~so~~ is denoted by

$$Acc(D)$$

Definition: Let $f: D \longrightarrow \mathbb{R}$, $\xi \in Acc(D)$. We say that f tends to $l \in \mathbb{R} \cup \{\pm\infty\}$ if for every sequence $(x_n) \subseteq D \setminus \{\xi\}$ s.t. $x_n \longrightarrow \xi$, $\lim_{n\to\infty} f(x_n) = l$

$\left(y_n = f(x_n) \quad \lim y_n = l \right)$



f

ξ

Example: $f(x) = x$ $\quad f: \mathbb{R} \to \mathbb{R}$

$\xi \in Acc(\mathbb{R}) = \mathbb{R} \cup \{\pm\infty\}$

$$\lim_{x \to \xi} f(x) = \lim_{x \to \xi} x = \xi$$

$\forall \; x_n \to \xi \qquad f(x_n) \to l = \xi$

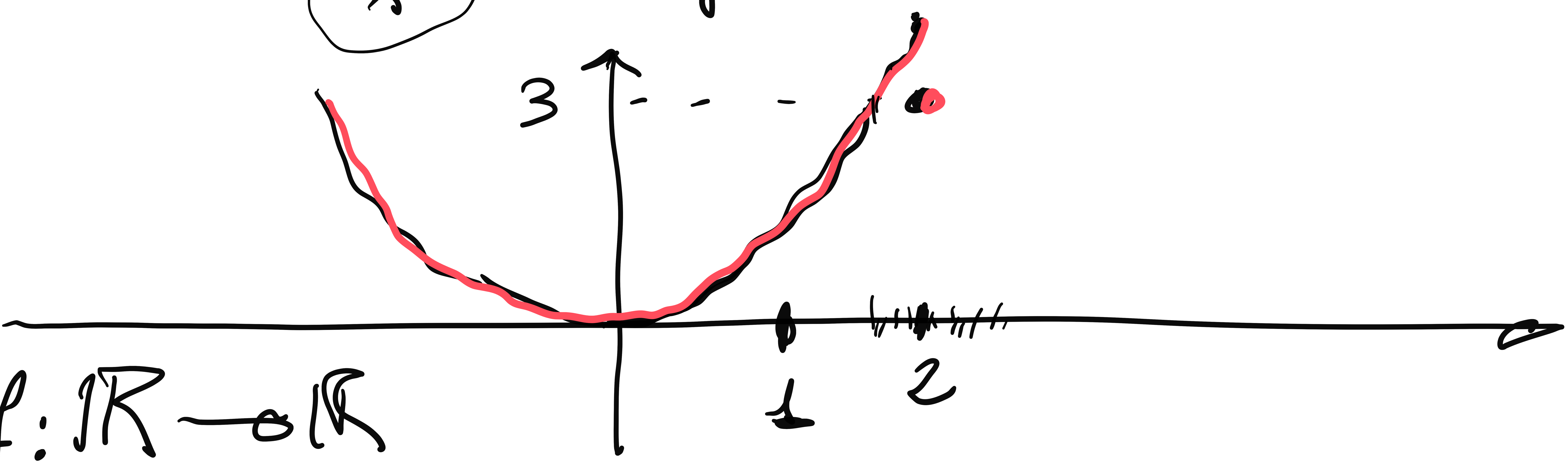
$\qquad\qquad\qquad\qquad\; x_n \to \xi$

Example: $f(x) = \text{constant} = C \in \mathbb{R}$

$\xi \in \mathbb{R} \cup \{\pm\infty\} \qquad x_n \to \xi$

$\qquad\qquad f(x_n) = C \to C$

$$f(x) = \begin{cases} 3 & \text{if } x = 2 \\ 3x & \text{if } x > 2 \\ x^2 & \text{if } x < 2 \end{cases}$$



$f: \mathbb{R} \to \mathbb{R}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 \underset{\text{check}}{=} 1$$

$\xi < 2$ $\quad \lim_{x\to\xi} f(x) = \lim_{x\to\xi} x^2 = \xi^2$

$x > 2$ $\quad \lim_{x\to\xi} f(x) = \lim_{x\to\xi} 3x \underset{\text{check}}{=} 3\xi$

$\xi = 2$ $\quad x_n \to 2 \quad x_n > 2 \quad \lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} 3x_n =$

$= 3\cdot 2 = 6.$

f is continuous on $\mathbb{R}\setminus\{2\}$

$x_n \to 2 \quad x_n < 2 \quad \lim f(x_n) = \lim x_n^2 =$

$= 4$

two sequences give different limit

$\nexists \lim_{x\to 2} f(x)$

Definition $f: D \to \mathbb{R}$, $\xi \in D$

$\xi \in Acc\,(D \cap [\xi, +\infty[)$ $\quad (D \cap]-\infty, \xi])$

$\lim_{x\to\xi^{\pm}} f(x) = \ell$ $\quad$ ("right limit") ("left" limit)

if $\forall (x_n) \subset D \cap [\xi, +\infty[\setminus \{\xi\}$ $\quad D \cap]-\infty, \xi] \setminus \{\xi\}$

$\lim_{n\to\infty} f(x_n) = \ell$

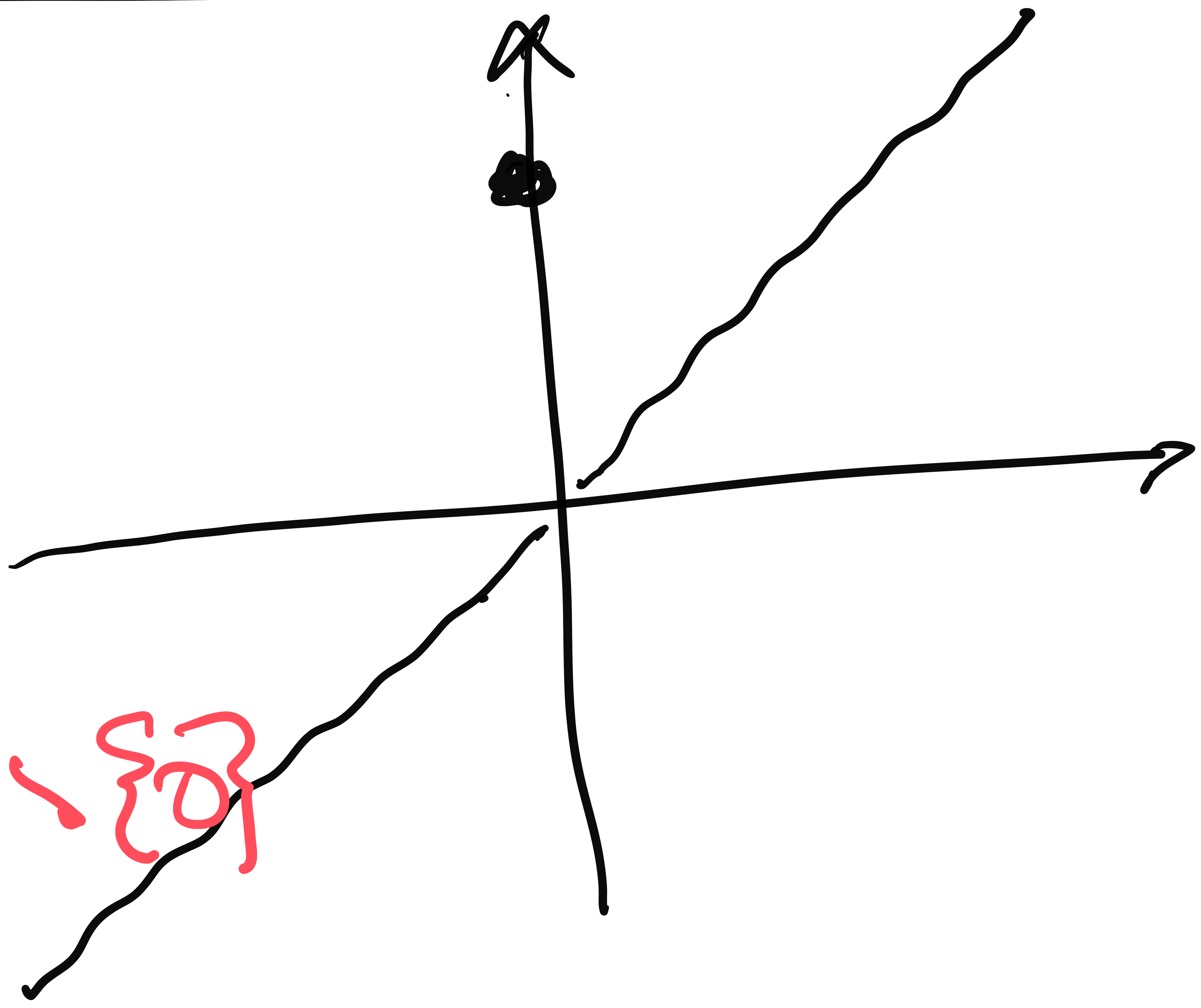
Theorem: $f: D \to \mathbb{R}$, $\xi \in D$

$\xi \in (Acc\, D \cap [\xi, +\infty[) \cap (Acc\, D \cap]-\infty, \xi])$

$\exists \lim_{x \to \xi} f(x) = l \iff \exists \lim_{x \to \xi^+} f(x) = l$
and
$\lim_{x \to \xi^-} f(x) = l$

Prove it by exercise.

Example



$f(x) = x \quad x \neq 0$

$f(x) = 3 \quad x = 0$

f is continuous on $S = \mathbb{R} \setminus \{0\}$

$\lim_{x \to 0} f(x) = 0$

$\left(\text{but } \lim_{x \to 0} f(x) \neq f(0) = 3 \right)$

<u>Definition:</u> $f : D \to \mathbb{R}$

$\xi \in D \cap Acc(D)$ if

$$\lim_{x \to \xi} f(x) = f(\xi)$$

we say that f is <u>continuous at ξ</u>

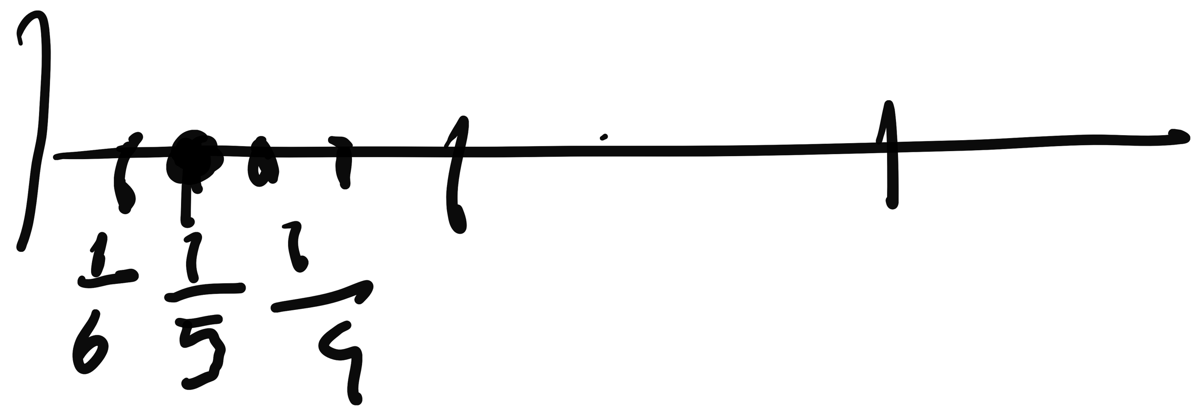
$S \subset D$

If f is continuous at every $\xi \in S$

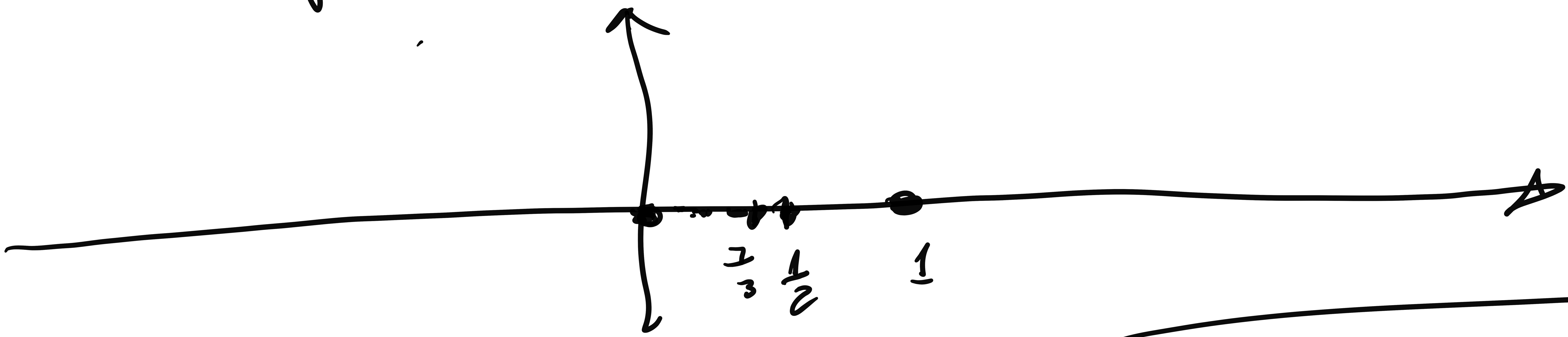
we say f is <u>continuous on S</u>

and we write $f \in \mathcal{C}(S)$

$$f : \underbrace{\left\{\frac{1}{n},\ n \in \mathbb{N}\right\} \cup \{0\}}_{D} \longrightarrow \mathbb{R}$$



$$f(x) = \sin x$$



$$Acc(D) = \{0\}$$

$\boxed{\text{Every } \xi \neq 0 \text{ is NOT an acc. point}}$

$$\lim_{x \to 0} \sin(x) = 0$$

because: (x_n) $\quad x_n \to 0 \quad x_n \in D \setminus \{0\}$

$$\lim_{n \to \infty} (x_n) : \lim_{n \to \infty} \sin(x_n) \to 0$$

$$\underset{\downarrow\, 0}{0} \le \underset{\downarrow\, 0}{\sin(x_n)} \le \underset{\downarrow\, 0}{x_n}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin x = 0 = \sin(0) = f(0)$$

$\Longrightarrow$ f is continuous at zero.

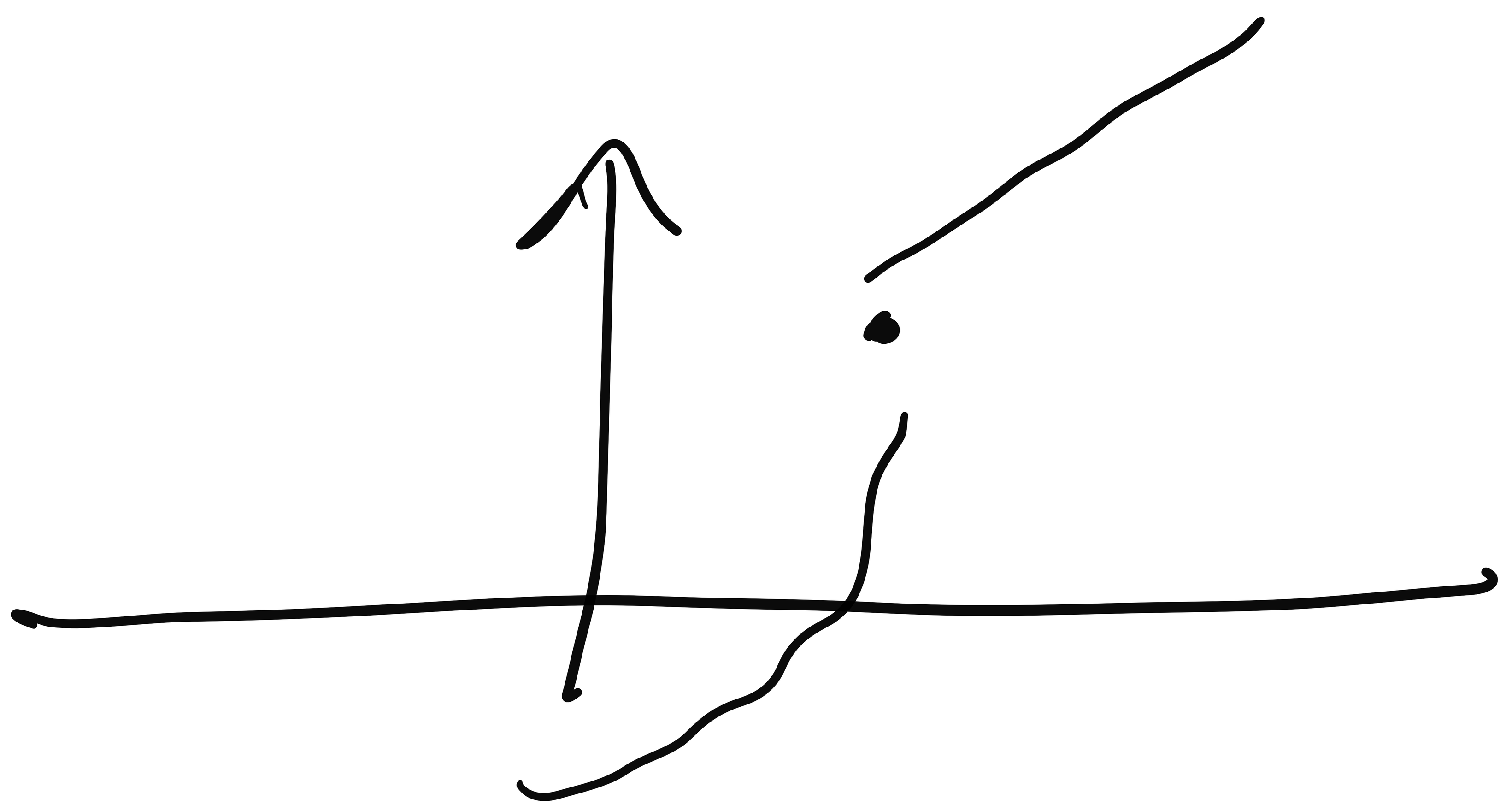
Definition: $f: D \longrightarrow \mathbb{R}$

$\xi \in Acc(D)$. I can prolong (or change) the function in ξ and obtain a cont. func $\hat{f}$ in ξ $\Longleftrightarrow \lim_{x \to \xi} f(x) = l$

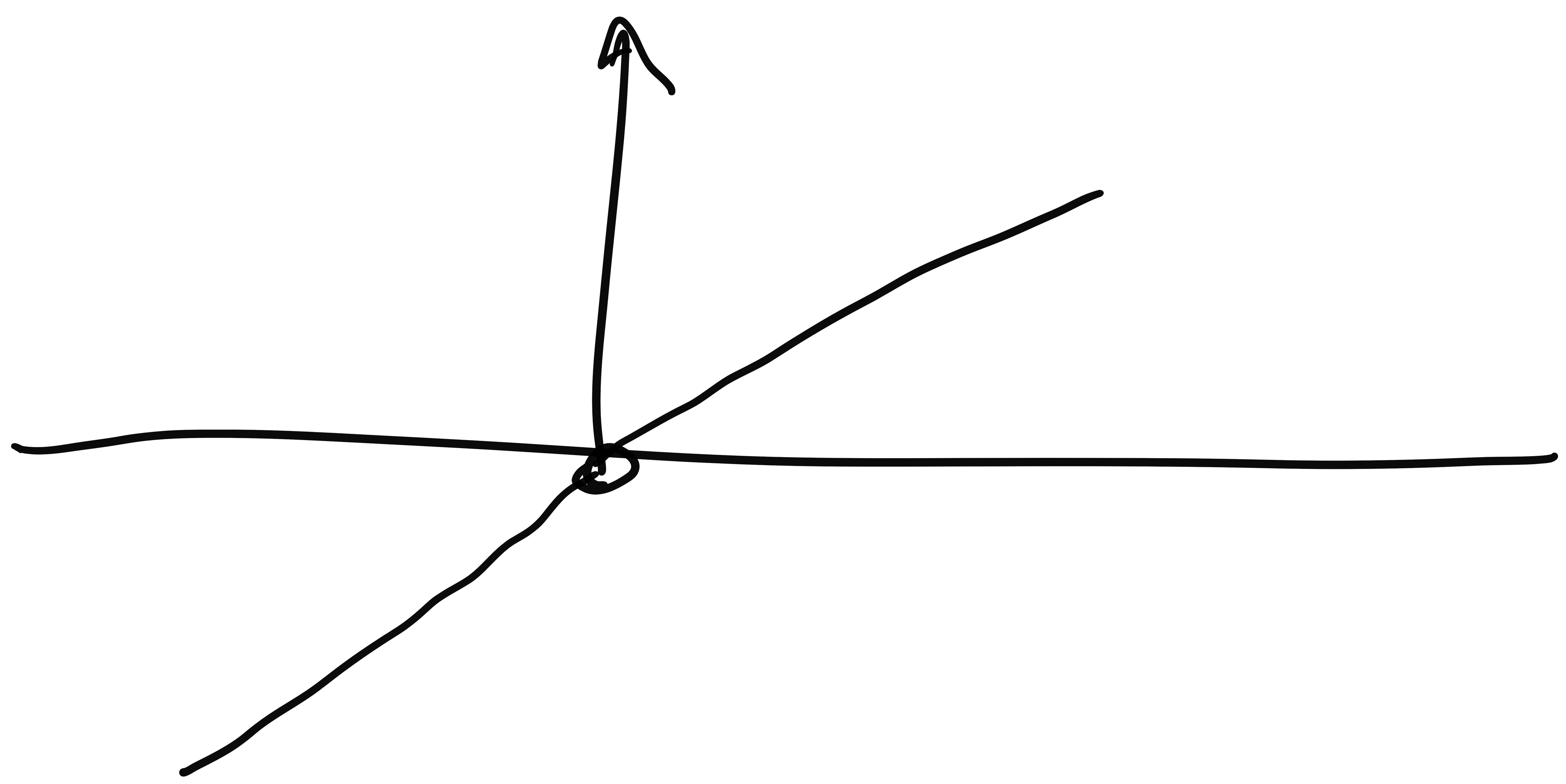
It is sufficient to set

$$\hat{f}(x) = \begin{cases} f(x) & x \notin \xi \\ l & x \in \xi \end{cases}$$

Ex



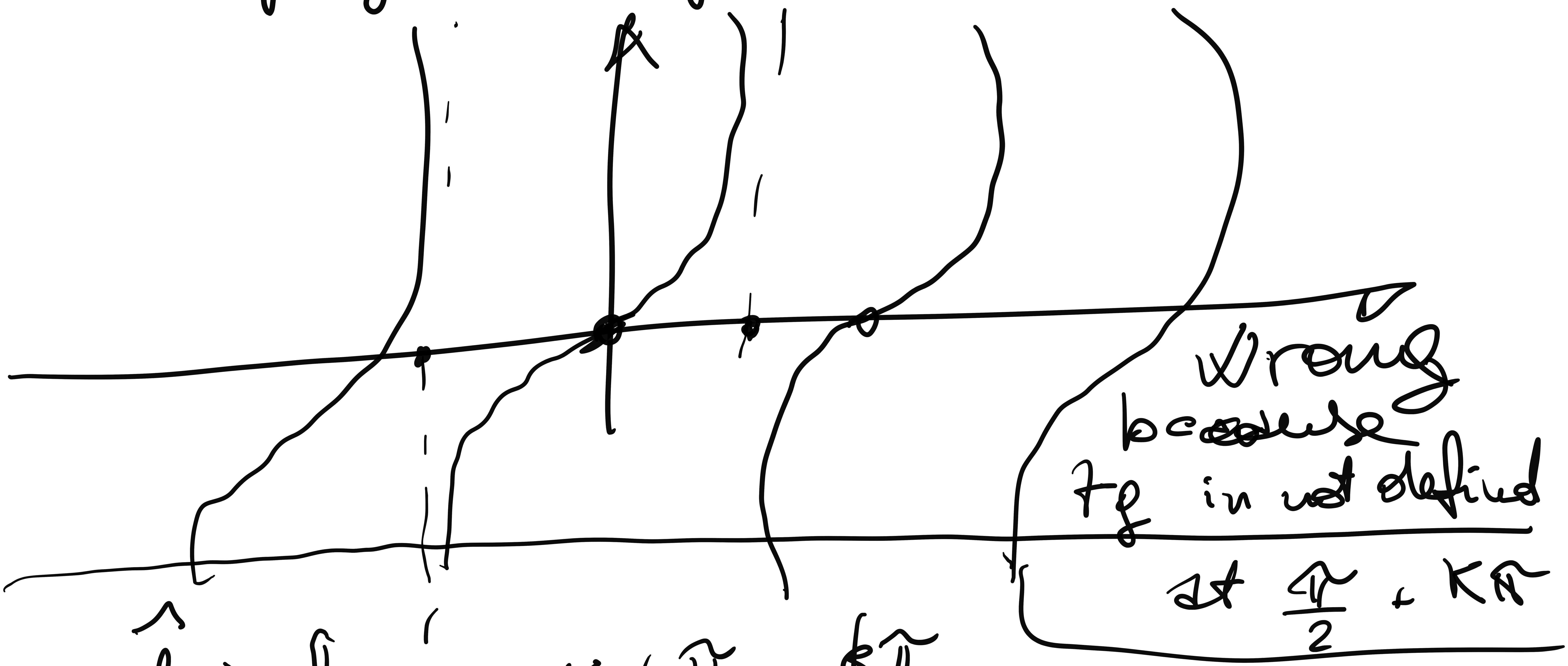
NOT changeable in a cont fun



yes

Theorem: All powers x^α $(\alpha \in \mathbb{R})$ are continuous on their domains, exponential b^x are continuous on their domains trigonometric function are cont. (on their domain)

<u>Wrong objection</u>: tg is not continuous because you cannot draw the graph without leaving the page at points:



$$\hat{f}(x)=\begin{cases} tg\, x & x\neq \frac{\pi}{2}+k\pi \\ \dots & \forall x=\frac{\pi}{2}+k\pi \end{cases}$$

doesn't exist a continuous $\hat{f}$ at those point (because limits don't exist)

$\xi \in \mathbb{R} \quad l \in \mathbb{R} \qquad f : D \longrightarrow \mathbb{R}$

$$\lim_{x \mapsto \xi} f(x) = l$$

if $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$\forall x \in [\xi - \delta, \xi + \delta] \cap D \setminus \{\xi\}$

$l - \varepsilon \leq f(x) \leq l + \varepsilon$