Theorem:

$a_n \longrightarrow l_1$

$b_n \longrightarrow l_2$

1) $l_1 \in \mathbb{R}$ $l_2 \in \mathbb{R}$

$$a_n + b_n \longrightarrow l_1 + l_2$$

2) $a_n b_n \longrightarrow l_1 \cdot l_2$

3) $\dfrac{a_n}{b_n} \longrightarrow \dfrac{l_1}{l_2}$

if $l_2 \neq 0$.

(if $\underset{(5)}{l_2 > 0} \Rightarrow \exists N \; \forall n > N$

$\underset{(5)}{b_n > 0}$

Property: If $c_n \longrightarrow l \in \mathbb{R}$

$\Longrightarrow \{c_n, n \in \mathbb{N}\}$ is bounded (i.e. $\exists K > 0$ s.t. $|c_n| < K$

$\Updownarrow$

$-K < c_n < K$)

Proof: $\varepsilon = 1$ $\exists N$

$\forall n \geq N$ $\underbrace{l - 1 < c_n < l + 1}$

$M = \max\{|l| + 1, \underbrace{|c_1|, \dots, |c_{N-1}|}\}$

$-M \leq c_n \leq M$ $\forall n \in \mathbb{N}$

Theorem $a_n \to \ell_1 \in \mathbb{R}$
$b_n \to \ell_2 \in \mathbb{R}$

$\Longrightarrow$ $a_n \cdot b_n \to \ell_1 \cdot \ell_2$

Proof $\forall \varepsilon > 0$

$\exists N_1 : |a_n - \ell_1| < \frac{\varepsilon}{2A}$

$\exists N_2 : |b_n - \ell_2| < \frac{\varepsilon}{2A}$

$$|\ell_1\ell_2 - a_n b_n| =$$

$$|\underbrace{\ell_1\ell_2 - a_n\ell_2} + \underbrace{a_n\ell_2 - a_n b_n}|$$

$$\leq |\ell_1\ell_2 - a_n\ell_2| + |a_n\ell_2 - a_n b_n|$$

$$= |\ell_2||\ell_1 - a_n| + |a_n||\ell_2 - b_n|$$

We know that $\exists\, M>0$ such that $|a_n| < M \ \forall n \in \mathbb{N}$

$$\leq |\ell_2|\,|\ell_1 - a_n| + M\,|\ell_2 - b_n| \leq$$

$$A = (|\ell_2| + M)$$

$$\leq A\left(\underbrace{|\ell_1 - a_n|}_{<\frac{\varepsilon}{2A}} + \underbrace{|\ell_2 - b_n|}_{<\frac{\varepsilon}{2A}}\right)$$

$$\leq \cancel{A}\left(\frac{2\varepsilon}{\cancel{2A}}\right) = \boxed{\varepsilon}$$

$$|a_n b_n - \ell_1 \ell_2| \leq \varepsilon$$

$$\forall n \geq N := \max\{N_1, N_2\}$$

What about infinite limits?

How do they behave with operations?

Case 1 $a_n \to l_1 \in \mathbb{R}$

$b_n \to +\infty$

$a_n + b_n \to +\infty$

Proof $\forall C+K \quad \exists N_1 \in \mathbb{N}$ s.t.

$\forall n \geq N \quad b_n > C+K$

$a_n \to l_1 \Longrightarrow (a_n)$ is bounded

$\Longleftrightarrow \quad |a_n| \leq K > 0 \quad \forall n \in \mathbb{N}$

We want to ~~prove~~

$\forall C \quad a_n + b_n \geq C \quad \forall n \geq N_1$

from hypothesis

$a_n + b_n \geq M - K \geq C$

choose $M = \underbrace{C + K}$

With same proof (a_n) is bounded

$b_n \to +\infty$

$a_n + b_n \to +\infty$

Case? $a_n \to \ell \in \mathbb{R}$ (or even less, i.e. a_n is bounded)

$b_n \to -\infty$

$$\Longrightarrow \quad a_n + b \to -\infty$$

$a_n \to +\infty$	$a_n \to -\infty$
$b_n \to +\infty$	$b_n \to -\infty$
$a_n + b_n = +\infty$	$a_n + b_n = -\infty$

$a_n \to +\infty$

$b_n \to -\infty$

$a_n + b_n =$

$a_n = n + 27 \to +\infty$

$b_n = -n \to -\infty$

$a_n + b_n = 27 \to 27$

$a_n = n + (-1)^n \to +\infty$

$b_n = -n$
$\downarrow$
$-\infty$

$a_n + b_n = \cancel{n} + (-1)^n - \cancel{n}$
$= (-1)^n$

$a_n = n^2$

$b_n = -n$

$a_n + b_n = n^2 - n =$
$= \underset{\downarrow \atop +\infty}{n} \underbrace{(n-1)}_{\downarrow \atop +\infty} \to +\infty$

$a_n = -n^2$

$b_n = n$

$a_n + b_n \to -\infty$

if $a_n \to +\infty$ $\quad b_n \to -\infty$

$a_n + b_n$ indetermined

$a_n \to +\infty$

$b_n \to l \gtrless 0$

$a_n \cdot b_n \to \pm\infty$

$a_n \to +\infty$

$b_n \to 0$

$a_n b_n$ is indetermined

$$a_n = n^7 \to +\infty$$
$$b_n = \frac{1}{n^8}$$
$$a_n \cdot b_n = \frac{1}{n} \to 0$$

$$a_n = n^8$$
$$b_n = \frac{1}{n^7}$$
$$a_n \cdot b_n = n \to +\infty$$

$$a_n = n^7$$
$$b_n = \frac{1}{n^7}$$
$$a_n \cdot b_n = 1$$

$$a_n \to \ell > 0$$
$$b_n \to 0$$
$$b_n \neq 0$$
$$\frac{a_n}{b_n}$$

We need more

$$b_n = \frac{(-1)^n}{n^2}$$

$$\frac{a_n}{b_n} = a_n n^2 (-1)^n$$

no limit

this changes sign at every steps

Def $\boxed{\lim_{n\to\infty} b_n = 0+}$

$\exists N \quad \forall n \geq N \quad b_n > 0$

$\lim_{n\to\infty} b_n = 0$

Example $\frac{1}{n}, \quad \frac{1}{n^2+2n-3} \quad \ldots$

Theorem $a_n \to l > 0$

$b_n \to 0+$

$\Longrightarrow \quad \frac{a_n}{b_n} \to +\infty$

$a_n \to +\infty$

$b_n \to -\infty$

$a_n \cdot b_n \to -\infty$

$a_n \to +\infty$

$b_n \to +\infty$

$a_n b_n \to +\infty$

and so on.

$a_n \to +\infty$

$b_n \to 0^+ \, (-)$

$\frac{a_n}{b_n} \to +\infty \, (-\infty)$

$a_n \to +\infty$

$b_n \to +\infty$

$\frac{a_n}{b_n}$

$$\frac{n^2 + \sin n + \sqrt{n}}{(n+3)^2} =$$

Numerator $\to +\infty$

Denominator $\to +\infty$

$$\Rightarrow \frac{n^2 + \sin n + \sqrt{n}}{n^2 + 6n + 9}$$

$$\frac{\cancel{n^2}\left(1 + \frac{\sin n}{n^2} + \frac{1}{n^{\frac{3}{2}}}\right)}{\cancel{n^2}\left(1 + \frac{6}{n} + \frac{9}{n^2}\right)}$$

$\left(1 + \frac{\sin n}{n^2} + \frac{1}{n^{\frac{3}{2}}}\right) \to 1$

$\left(1 + \frac{6}{n} + \frac{9}{n^2}\right) \to 1$

1

$$\frac{n^3 + \sin n + \sqrt{n}}{(n+3)^2} = \frac{n^{\cancel{3}}\underbrace{\left(1 + \frac{\sin n}{n^3} + \frac{1}{n^{\frac{5}{2}}}\right)}_{\to 1}}{\cancel{n^2}\underbrace{\left(1 + \frac{6}{n} + \frac{9}{n^2}\right)}_{\to 1}}$$

$$\longrightarrow +\infty$$

$$\frac{n + \sin n + \sqrt{n}}{(n+3)^2} =$$

$$\frac{\cancel{n}\underbrace{\left(1 + \frac{\sin n}{n} + \frac{1}{n^{\frac{1}{2}}}\right)}_{\to 1}}{n^{\cancel{2}}\underbrace{\left(1 + \frac{6}{n} + \frac{9}{n^2}\right)}_{\to 1}} \to 0$$

<u>Def</u> Suppose that (a_n), (b_n) have infinite limit.

(a_n) is a <u>higher order</u> infinity than b_n if

$$\lim_{n \to \infty} \frac{|a_n|}{|b_n|} = +\infty$$

(a_n) and (b_n) are of the <u>same order</u>

if $\lim\limits_{n\to\infty} \frac{|a_n|}{|b_n|} = C \neq 0$, $C \in \mathbb{R}$

(a_n) is <u>asymptotic</u> to (b_n)

if $\lim\limits_{n\to\infty} \frac{|a_n|}{|b_n|} = 1$

If $a > 1$ $\alpha > 0$ $b > 1$

→ b^n is of higher order than n^a

→ n^a is of higher order than $\log_a n$

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notation in the NOTES

$$\log_a n \underset{\infty}{\ll} n^\alpha \underset{\infty}{\leq} b^n$$

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i.e. $\boxed{\lim \frac{b^n}{n^a} = +\infty}$ (*)

$\lim \frac{n^\alpha}{\log_a n} = +\infty$

$\left(\Rightarrow \lim \frac{b^n}{\log_a n} = +\infty \right)$

Prove $\boxed{\lim \frac{b^n}{n^\alpha} = +\infty}$

$0 < \alpha < 1$ $\qquad b = 1 + h$

$$b^n = (1+h)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} h^k \geq 1 + nh$$

$$\frac{b^n}{n^\alpha} = \frac{(1+h)^n}{n^\alpha} \geq \frac{1+nh}{n^\alpha} = \underbrace{\frac{1}{n^\alpha}}_{0} + h\, \overbrace{n^{1-\alpha}}^{>0} \to \infty$$

$\alpha \geq 1$

$$\left(\frac{b^n}{n^\alpha}\right) = \underbrace{\left(\frac{b^{\frac{n}{2\alpha}}}{n^{\frac{1}{2}}}\right)^{2\alpha}} = \underbrace{\left(\frac{\left(b^{\frac{1}{2\alpha}}\right)^n}{n^{\frac{1}{2}}}\right)}_{\to +\infty}{}^{2\alpha}$$

$\to +\infty$

$$\tilde{b} = b^{\frac{1}{2\alpha}} \qquad \frac{\tilde{b}^n}{n^{\frac{1}{2}}}$$

We have used

$c_n \to +\infty$

$c_n^\beta \to +\infty \quad \forall \beta > 0$ $\leftarrow$ prove by exercise