Università degli Studi di Padova

Complex Systems Engineering
Mathematical Physics
November, 12 – 2021
Time: 2:30 hours

Exercise 1. (a) Draw the phase portrait for the equation

$$\dot{z} = (z^2 - 1)z^2 + \mu \qquad z \in \mathbb{R}$$

depending on the parameter $\mu \in \mathbb{R}$.
(b) Draw the bifurcation diagram for the equilibria, specifying their (asymptotic) stability/instability.

Exercise 2. Let consider the system of differential equations

$$\begin{cases} \dot{x} = -x - y + x^2 \\ \dot{y} = x - y - xy \end{cases} \qquad (x, y) \in \mathbb{R}^2.$$

(a) What can you say about the instability of the origin by using the First Lyapunov method (Spectral method)?
(b) Can you use

$$W(x, y) = \frac{1}{2}(x^2 + y^2)$$

as Lyapunov function in order to study the stability of the origin? In case, what can you deduce?

Exercise 3. Let consider the system of differential equations

$$\begin{cases} \dot{x} = y^3 - 4x \\ \dot{y} = y^3 - y - 3x \end{cases} \qquad (x, y) \in \mathbb{R}^2.$$

(a) Show that the origin is a stable spiral for the linearized system. Moreover, determine the winding direction.
(b) Determine –motivating the answer– the (asymptotic) stability/instability of the origin for the nonlinear system.
(c) What does it mean that a subset is invariant for a given dynamics? Show that the line $x = y$ is invariant for the nonlinear system.

Exercise 4. Let consider a particle P of mass $m = 1$, constrained on the x-axis and subjected to conservative forces with potential

$$V(x) = -x^4 + 2x^2 - 2.$$

(a) Draw the phase portrait, specifying the quality of equilibria.
(b) Determine the set of initial conditions giving periodic solutions.
(c) If the particle is in $x = 0$ with velocity $v = \sqrt{2}$, how much time occur to arrive in $(1, 0)$? And if the initial velocity is $v = 2$? Motivate the answers.
(d) Why this dynamical system cannot have asymptotically stable equilibria?

Question 1. Give the definition of limit cycle and write a (simple) system of differential equations whose dynamics exhibits a limit cycle.

Question 2. Give the definition of first integral for a flow. Prove that for a mechanical system with Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\langle a(q)\dot{q}, \dot{q}\rangle - V(q)$$

the total energy is a first integral.