Lesson 12 - 24/10/2022

Ex 1 Let consider the diff. eq. $\ddot{x} = -V_K'(x)$, $x \in \mathbb{R}$.
where $V_K(x) = Kx(x^2 - K)$.
(a) Draw the phase-portrait for every $K \in \mathbb{R}$.
(b) Let $K = 1$. Establish for which values $v \in \mathbb{R}$ the solution with initial datum $(x(0), v(0)) = (0, v)$ is periodic.

Ex 2 • Let consider the diff. eq. $\dot{x} = Kx e^{-x^2/2}$, $x \in \mathbb{R}$ and $K \in \mathbb{R}$.
(a) Draw the phase-portrait for every $K \in \mathbb{R}$.
(b) Determine equilibria and their quality $\forall K \in \mathbb{R}$. Draw the bif. diagram.
(c) Linearize $\dot{x} = Kx e^{-x^2/2}$ around the origin.

• Let consider the diff. eq. $\ddot{x} = -V'(x)$, where $V(x) = x e^{-x^2/2}$ $(K=1)$.
(a) Draw the phase-portrait
(b) Determine equilibria and their quality.
(c) Determine for which values of $v \in \mathbb{R}$ the solution with initial datum $(-1/2, v)$ is periodic.
(d) By using an appropriate Lyap. function, discuss the stability of $(-1, 0)$ for $\ddot{x} = -V'(x) + F_\mu(x, v)$, where $F_\mu(x, v) = -2\mu(x+1)v^2$, $\mu > 0$.

Ex 3 Let $\begin{cases} \dot{x} = 4y(y^2 - 1) \\ \dot{y} = 4x(x^2 - 1) \end{cases}$

Verify that there exists a first integral for X and explicitly determine it.

Ex 4 Let $\begin{cases} \dot{x} = -x^3 + y^3 \\ \dot{y} = -x^3 - y^3 \end{cases}$

Prove that $(0,0)$ is a stable eq, by using one of these functions:

$f(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}(x^4 + y^4)$ NO!

$g(x, y) = \frac{1}{4}(x^4 + y^4)$

$h(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) + \frac{1}{4}(x^4 - y^4)^2$ No!

Ex 5 Let consider the 2nd order diff eq: $\ddot{x} = -\sin x - x^4 \dot{x}$, $x \in \mathbb{R}$.
Discuss the stability of $(0, 0)$ with both spectral and Lyapunov function methods.

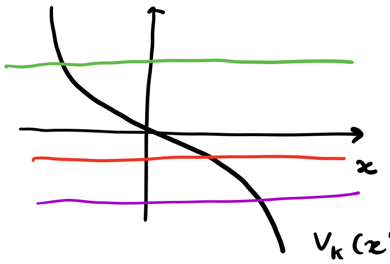
Remark: ATTRACTIVENESS (attracting fixed point/orbit) is impossible in conservative systems!

Ex 1

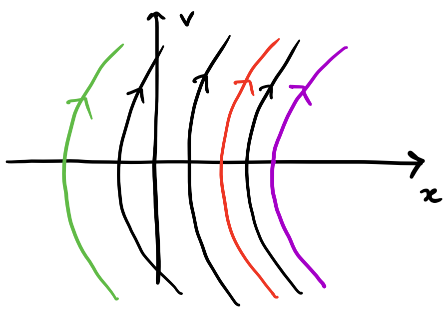
$V_K(x) = Kx(x^2 - K)$

- $K < 0 \Rightarrow Kx(x^2-K) = 0$ iff $x = 0$

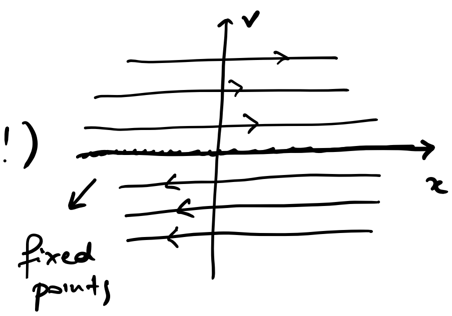
  Moreover $\lim_{x \to \pm\infty} V_K(x) = \mp\infty$



$V_K(x)$, for $K<0$



- $K = 0 \Rightarrow V_0(x) \equiv 0$ (free particle !)



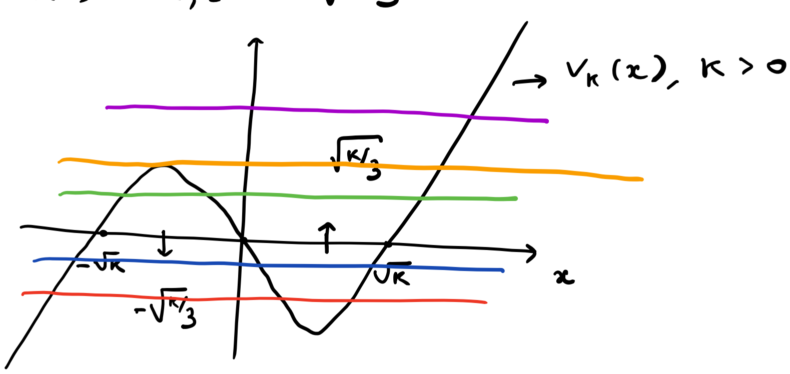
- $K > 0 \Rightarrow V_K(x) = Kx(x^2-K) = 0$

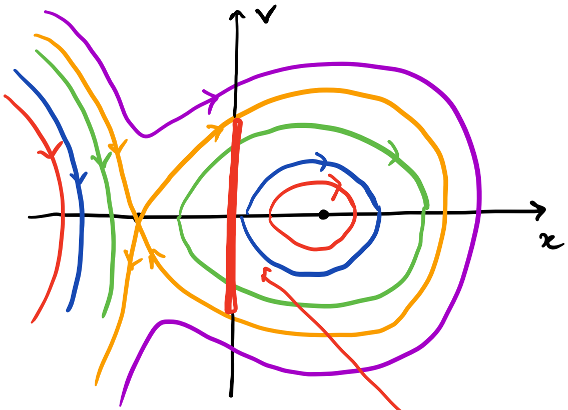
  $\Leftrightarrow x = 0, \quad x = \pm\sqrt{K}.$

  Moreover $\lim_{x \to \pm\infty} V_K(x) = \pm\infty$

  $V_K'(x) = K(x^2-K) + Kx \cdot 2x = 3Kx^2 - K^2 = 0$

  $\Leftrightarrow x_{1,2} = \pm\sqrt{K/3}$





energy on the separatrix for $K=1$

$\boxed{K=1} \to V_1(x) = x(x^2-1)$

The condition is: $E(0,v) = \frac{1}{2}v^2 + \underbrace{V_1(0)}_{\substack{\| \\ 0}} < \overbrace{V_1\left(-\frac{1}{\sqrt{3}}\right)}$

That is:

$\frac{1}{2}v^2 + 0 < \frac{2}{3\sqrt{3}} \iff v^2 < \frac{4}{3\sqrt{3}} \iff$

$$\boxed{v \in \left(-\frac{2}{\sqrt{3\sqrt{3}}}, +\frac{2}{\sqrt{3\sqrt{3}}}\right)}$$

The other condition:

$E(0,v) = \underbrace{\frac{1}{2}v^2} + V_1(0) \geq V_1\left(\frac{1}{\sqrt{3}}\right)$ is automatically satisfied,

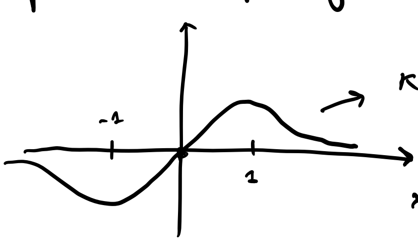
since $\frac{1}{2}v^2 \geq 0$ and $V_1\left(\frac{1}{\sqrt{3}}\right) < 0$

$\boxed{\text{Ex 2}}$ First part —x —x —

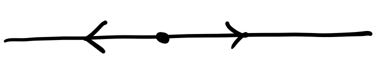
$\dot{x} = Kx\, e^{-x^2/2}$, $K \in \mathbb{R}$, $x \in \mathbb{R}$.

a) Phase portrait, depending on $K \in \mathbb{R}$

• $K > 0$



$\to Kx\, e^{-x^2/2}$, for $K > 0$

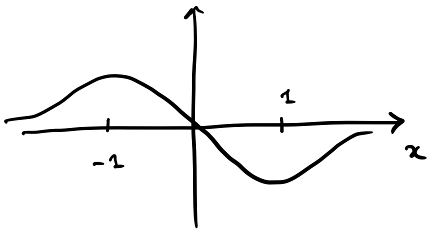


0 = repeller

• $k=0$

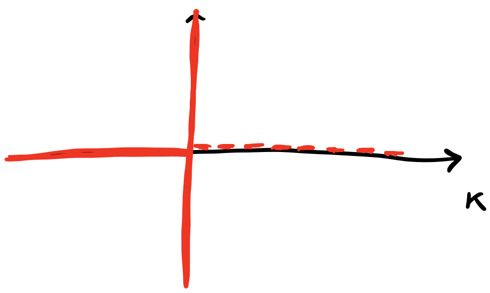
$\mathbb{R}$ = fixed points.

• $k<0$



0 = attractor.

b)



--- = unstable
— = stable

c) Linearization around $x=0$.

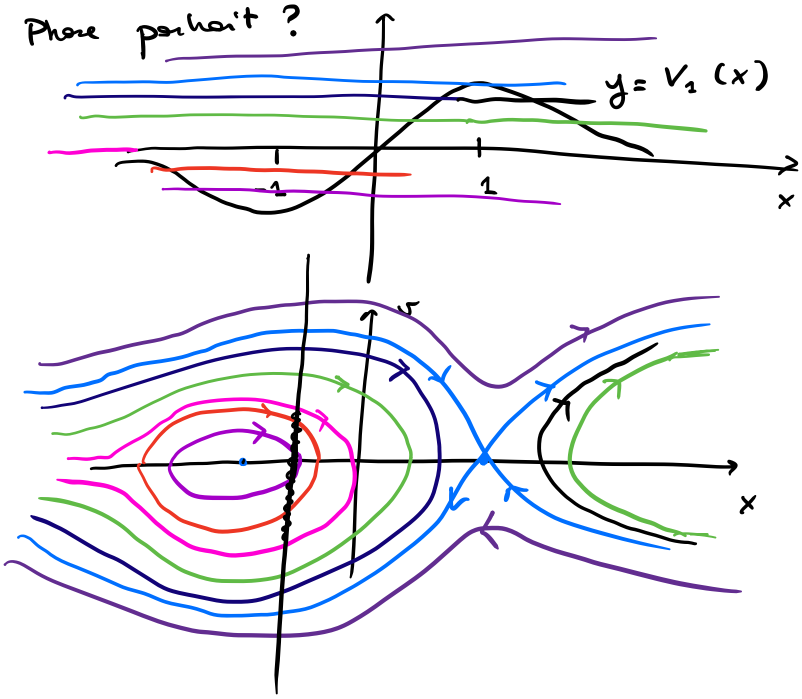
$$V_k'(x) = k e^{-x^2/2}(1-x^2)$$

$$V_k'(0) = k \quad \Rightarrow \quad \dot{x} = kx$$

Second part

(a) $\ddot{x} = -V'(x)$ where $V(x) = xe^{-x^2/2}$

Phase portrait ?



A detail Suppose $E \geq 0$ (No periodic orbits)

$$\frac{1}{2}v^2 + x e^{-x^2/2} = E \Leftrightarrow v^2 = 2E - 2x e^{-x^2/2}$$

$$\Rightarrow \lim_{x \to \pm\infty} v(x) = \lim_{x \to \pm\infty} \sqrt{2E - 2x e^{-x^2/2}} = \pm\sqrt{2E}$$

(b)

Equilibria : $(-1, 0) \to$ stable.

$(+1, 0) \to$ unstable.

(c) Condition so that the solution starting from $(-1/2, v)$ is periodic is

$$E(-1/2, v) = \frac{1}{2}v^2 - \frac{1}{2}e^{-1/8} \underbrace{< 0}_{\text{condition!}}$$

$$\Leftrightarrow v^2 < e^{-1/8}$$

$$\Leftrightarrow v \in \left(-e^{-1/16}, e^{1/16}\right)$$

(d) $\ddot{x} = -V'(x) + F_\mu(x, v)$ $\quad \mu > 0$

$$F_\mu(x, v) = -2\mu(x+1)^2 v$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -V'(x) - 2\mu(x+1)^2 v \end{cases}$$

$(-1, 0) \to$ Equilibrium. Remains stable?

$$E(x, v) = 1/2\, v^2 + V(x) - V(-1)$$
$$= 1/2\, v^2 + x e^{-x^2/2} - e^{-1/2}$$

1st. Cond. of Lyapunov theo : ok

Now we check the 2nd one.

$$L_X E(x, v) = \nabla E(x, v) \cdot X(x, v)$$
$$= V'(x)\dot{x} + v\dot{v} =$$
$$= \cancel{V'(x) v} + v\left(\cancel{-V'(x)} - 2\mu(x+1)^2 v\right)$$

$$= -2\mu (x+1)^2 v^2 \leq 0 \quad \text{near } (-1,0) \Rightarrow \text{stable!}$$

—x—x—

$\boxed{\text{EX 3}}$

$$\begin{cases} \dot{x} = 4y(y^2-1) \\ \dot{y} = 4x(x^2-1) \end{cases}$$

$f(x,y)$ is a first integral iff $L_X f(x,y) \equiv 0$ ($f \circ \phi_t(x,y) = f(x,y)$ $\forall t \in \mathbb{R}$ when f is only continuous)

$L_X f(x,y) \equiv 0$ means

$$\nabla f(x,y) \begin{pmatrix} 4y(y^2-1) \\ 4x(x^2-1) \end{pmatrix} \equiv 0$$

$$\Leftrightarrow \frac{\partial f}{\partial x}(x,y)\, 4y(y^2-1) + \frac{\partial f}{\partial y}(x,y)\, 4x(x^2-1) \equiv 0$$

we search for $f(x,y)$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x(x^2-1) \\ \dfrac{\partial f}{\partial y} = -4y(y^2-1) \end{cases}$$

$$\Leftrightarrow f(x,y) = (x^2-1)^2 - (y^2-1)^2 + c \quad (c \in \mathbb{R})$$

$\boxed{\text{EX4}}$

$$L_X g(x,y) = \nabla g(x,y) \cdot X(x,y) =$$

$$= x^3 \dot{x} + y^3 \dot{y} = x^3(-x^3 + \cancel{y^3}) + y^3(-\cancel{x^3} - y^3) =$$

$$= -x^6 - y^6 < 0 \quad \forall (x,y) \neq (0,0)$$

$\Rightarrow (0,0)$ is A.S. !

$\boxed{\text{EX 5}}$ $\ddot{x} = \underbrace{-\sin x}\ \underline{-x^4 \dot{x}}$

Stability of $(0,0)$ with both Lyapunov methods.

$$\begin{cases}\dot x = v\\ \dot v = -\sin x - x^4 v\end{cases} \qquad (\ddot x = -\sin x = -V'(x),\ V(x) = -\cos x)$$

- Spectral method $J(x,v) = \begin{pmatrix}0 & 1\\ -\cos x - 4x^3 v & -x^4\end{pmatrix}$

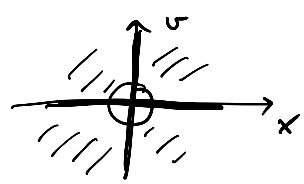
$$J(0,0) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \to \lambda_{1,2} = \pm i$$

$\to$ NO CONCLUSIONS on the non-linear system.
(Elliptic eq. of the linearization).

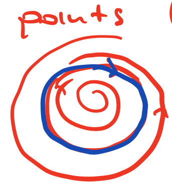
- $E(x,v) = \dfrac{v^2}{2} - \cos x + \underbrace{1}_{=-V(0)}$

$$L_X E(x,v) = \nabla E(x,v)\cdot X(x,v) = (\sin x)\,\dot x + v\,\dot v = \cancel{\sin x\,(v)} + v(\cancel{-\sin x} - x^4 v) = \underline{-x^4 v^2} \leq 0$$

$\Rightarrow$ $(0,0)$ is stable!



Remark

ATTRACTIVENESS (attracting fixed points  /
attracting orbits )
is impossible in conservative systems!

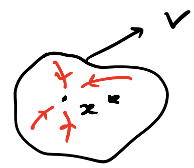
$\dot x = X(x)$, $x \in \mathbb{R}^n$.
A first integral is a function $f \in C^1(\mathbb{R}^n; \mathbb{R})$ s.t. $L_X f \equiv 0$. To avoid trivial functions, we also require that $f(x)$ be non-constant on every open set of $\mathbb{R}^n$.

(otherwise, every v.f. has a first integral, given by a (trivial) constant function)

$\Downarrow$

A conservative system cannot have attracting fixed points / orbits.

<u>Proof</u> Suppose – on the contrary – that $x^* \in \mathbb{R}^n$ is an attracting fixed points. Then $x^* \in V =$ = basin of attraction.



$$f(x^*) = f(x) \quad \forall\, x \in V$$

$f = E$ = total conserved quantity.

$f \equiv$ constant on V. ↯

$\boxtimes$

$$\lim_{t \to +\infty} \varphi_t(x) = x^*$$

and we know that f is constant along trajectories $\Rightarrow$ $f(\varphi_0(x)) = f(x) =$

$$= f\left(\lim_{t\to+\infty} \varphi_t(x)\right) =$$

$$= \lim_{t\to+\infty} \left(f(\varphi_t(x))\right) =$$

$$= f(x^*).$$

— x — x — x —